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Courant Institute of
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Magneto-Fluid Dynamics Division

The Stability of a Particular MHD Equilibrium with Flow

Eliezer Hameiri

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ABSTRACT

This report investigates the stability properties of a screw pinch equilibrium with flow. The question of local stability is fully answered by the derivation of a necessary and sufficient criterion. Unlike the static case, local modes can be suppressed by global properties of the plasma, as well as by a large sheared flow. When local instability does occur, the accumulation point of the unstable modes is found to be the point of overlap of the Alfvén and cusp continua.

The last chapter of the work develops a sufficient (but not necessary) condition for stability. Some bounds on the growth rates of instabilities are also obtained.

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Introduction

The term "stability" in the context of magnetohydrodynamic systems is usually used to describe a situation where small perturbation of a given equilibrium state, does not result in large change of the system. A simplified version of this concept is linear stability. It is assumed that the disturbances are so small that a linearized version of the equations of motion is sufficient to describe the system, and for stability it is required that solutions of the linearized equations do not grow unboundedly in time. Mathematical analysis is simplified if we ignore modes that do not grow exponentially in time. We adopt this common practice here.

The model we use in this thesis is ideal MHD, in which plasma is described as a non dissipative fluid, acting under the influence of hydrodynamic and electromagnetic forces. We use one of the simplest configurations for containment of plasma - a diffuse screw pinch, in which the plasma occupies the volume of a cylinder and is confined by the current flowing within it. Our model allows for an equilibrium state with steady flow, and the investigation of this state is what distinguishes this work from others previously done. A quick review of these works would provide us with a deeper insight into the difficulties involved.

The question of stability of MHD equilibria without flow was answered in principle by Bernstein et al [1]. These authors used an energy principle which provided a necessary and sufficient condition for exponential stability. The necessity part was proven by Laval, Mercier and Pellat [20]. The energy principle was worked out in a straightforward way by Newcomb [11] to provide a marginal stability criterion for the static screw pinch equilibrium. Goedbloed [21] arrived at similar results using a different argument based on oscillation theorems. The success in solving the static problem was made possible by the availability of a self adjoint formulation of the problem. The general solution to the equation $u_{tt} + Hu = 0$, where u belongs to some Hilbert space and H is a self adjoint operator in that space, is given by the linear combination $u = ae^{i\sqrt{H}\,t}u_1 + be^{-i\sqrt{H}\,t}u_2$ where the exponential operators are defined as the appropriate integral taken over the spectrum of H and u_1, u_2 are vectors in the space. Clearly H must be positive for exponential stability. This demonstration will convince the reader of the usefulness of the self adjoint formulation (although H in reference 1 was only shown to be formally self adjoint).

When flow is introduced into the system, the self adjointness is lost and there is a jump in the degree of difficulty in treating the case. Only a few steady

equilibria were investigated prior to this time. Chandrasekhar [22] and Newcomb [23] found stability criteria for purely rotational flows with a unidirectional magnetic field, and with respect to azimuthally symmetric perturbations only. Spies [24] recently worked out the stability of the θ-pinch (axial magnetic field) with solid rotational flow and with respect to axially independent perturbations.

In this work we consider equilibria with magnetic fields having toroidal as well as poloidal components, and flows having axial and azimuthal components. The price we pay for this generality is in being unable to conclude categorically whether a system is stable or not, but we have to focus our attention on parts of the spectrum.

In chapter II we investigate the continuous spectrum corresponding to the problem, and find that it consists of real members only and thus does not cause exponential instability. Another part of the spectrum that is relatively easy to handle is the accumulating sequences of eigenmodes, known as local (Suydam) modes. The investigation of such unstable modes requires the development of an asymptotic expansion technique for a non-standard case of a singular perturbation. This is carried out in chapter III and

appendix A. It is then found in chapter IV that, like the static case, unstable eigenmodes can only accumulate at the point of overlap of the various continuous spectra. We derive a necessary and sufficient condition for local stability, which unlike Suydam's condition for the static case, is not entirely local and depends on global properties of the plasma which may suppress the local modes. Another stabilizing effect is generated by the shear of the equilibrium flow. A large enough flow shear is sufficient for local stability.

In chapter V a Lagrangian description of the motion is used to obtain some bounds on the growth rates of unstable modes and we develop a sufficient condition for stability, which reduces to Newcomb's [11] necessary and sufficient condition in the static case.

CHAPTER I: Basic Equations

I.1 The Eulerian Equations of Motion

The equations which describe an ideal MHD (i.e. perfectly conducting) plasma are:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{u} = 0$$

$$\rho \frac{d\underline{u}}{dt} = - \operatorname{grad} p + \underline{j} \times \underline{B}$$

$$\frac{dS}{dt} = 0$$

(MF)

$$\frac{\partial \underline{B}}{\partial t} + \operatorname{curl} \underline{E} = 0$$

$$\operatorname{div} \underline{B} = 0$$

$$\operatorname{curl} \underline{B} = \underline{j}$$

$$\underline{E} + \underline{u} \times \underline{B} = 0$$

$$p = f(\rho, S)$$

where ρ, p, S, $\underline{u}$, $\underline{E}$, $\underline{B}$, $\underline{j}$ are the plasma mass density, pressure, entropy, velocity, the electric and magnetic fields and the current density. The symbol $\frac{d}{dt}$ stands for the Lagrangian time derivative $\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}$.

The first three equations express the conservation of mass, momentum and entropy. The next three equations are Maxwell's equations for the electromagnetic field with the displacement current omitted. The last two equations are Ohm's law for a perfectly conducting fluid and the given equation of state. The conditions for the applicability of these equations are discussed in reference 1.

We use the last three equations to eliminate $\underline{E}, \underline{j}$ and S from the system. In particular, we multiply the first equation by $\frac{\partial p}{\partial \rho}$ and the third one by $\frac{\partial p}{\partial S}$. Adding these two equations yields an equation for p which replaces the adiabatic law. System (MF) takes now the simplified form of the Lundquist equations [2]:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{u} = 0$$

$$\frac{dp}{dt} + \rho a^2 \operatorname{div} \underline{u} = 0$$

(L)

$$\rho \frac{d\underline{u}}{dt} = - \operatorname{grad}(p + \frac{1}{2}B^2) + (\underline{B}\cdot\underline{\nabla})\underline{B}$$

$$\frac{\partial \underline{B}}{\partial t} + \operatorname{curl}(\underline{B}\times\underline{u}) = 0$$

$$\operatorname{div} \underline{B} = 0$$

where $a^2 = \frac{\partial p}{\partial \rho}$ is the square of the speed of sound. It is well known that the last equation can be considered an initial

condition, since it will be satisfied for all time if satisfied initially. We point out that system(L) is hyperbolic [3], a useful property for our work.

In this treatise we are interested in a special plasma configuration: the diffuse screw-pinch with flow. The following assumptions characterize the steady state:

i. The plasma occupies the volume of a cylinder of length L (along the z axis, say) surrounded by a perfectly conducting wall.

ii. All quantities (the pressure, magnetic field, etc.) depend on the distance from the axis only (r - in cylindrical coordinates).

iii. The equilibrium magnetic field, and consistently the flow velocity, have vanishing radial components.

Such a flow has to satisfy only one non trivial equation of the system (L)

$$(1) \qquad (p + \frac{1}{2}B^2)' + \frac{B_\theta^2}{r} = \rho\frac{u_\theta^2}{r}$$

where the prime denotes differentiation with respect to r and subscripts denote vector components in cylindrical coordinates. Since the axial velocity does not enter this equation, one equilibrium state gives rise to many others by simply changing u_z.

The boundary conditions that have to be imposed consistently with the cylindrical geometry, are

i. Periodicity in z with a period L.

(BC) ii. $B_r = u_z = 0$ at the outer wall.

iii. Regularity at the origin.

These conditions apply of course to steady, as well as to time-dependent flows. The second condition is automatically satisfied by the steady state.

I.2 The Perturbed Equations

We turn now to the stability analysis of the system (L). Rather than working with the non-linear system, we restrict our attention to the question of linear stability of this system. Accordingly we consider a small perturbation of the equilibrium state, and linearize the system(L) about it. The coefficients of the linearized equations then depend on r only, and it is possible to Fourier transform the system with respect to the axial and angular variables, and to Laplace transform it in time. This is equivalent to assuming a dependence of every perturbed quantity $\hat{\xi}$, of the form

$$\hat{\xi}(r,\theta,z,t) = \xi(r) \exp i(m\theta+kz+\omega t).$$

Application of the Laplace transform is justified by the hyperbolocity of the system (L), from which we conclude that the linearized system is of a similar type and thus possesses only solutions that grow at most exponentially fast in time.

The distinction between the Laplace and Fourier transforms is made in order to underline the existence of non-homogeneous terms originating from initial conditions. These terms play an important role in identifying the continuous spectrum. However, since our main interest is

the point spectrum, we actually omit such terms.

On linearization about the screw pinch with flow as the steady state, the system (L) can be reduced to a 2×2 system of first order ordinary differential equations in only two of the perturbed quantities. All other perturbed quantities can be expressed in terms of these two. This substantial simplification is made possible by the choice of the screw pinch as our equilibrium state, which in turn makes the linearized system (L) algebraic (and not differential) in six of the eight dependent variables. The reduced system can be written as

$$(A') \qquad \sigma AC\begin{pmatrix} u \\ ip_* \end{pmatrix}' + \Lambda^o \begin{pmatrix} u \\ ip_* \end{pmatrix} = 0$$

where $u=u_r$ is the radial component of the perturbed flow, and $p_* = p+\underline{B}_o \cdot \underline{B}$ is the perturbed total pressure. All other perturbed quantities can be expressed as linear combinations of u, u', p_*. The explicit dependence is described in Appendix B.

The coefficients of system (A') are functions of the unperturbed flow. To avoid confusion, an equilibrium quantity will carry a subscript o when necessary. We use the following definitions:

$$\sigma = \omega+\underline{k}\cdot\underline{u}$$

$$A = H^2-\rho\sigma^2$$

$$C = B_*^2(H^2-\rho\sigma^2)$$

and we use the notation

$$\underline{k} = (0, \frac{m}{r}, k) \text{ in cylindrical coordinates.}$$

$$K^2 = k^2 + \frac{m^2}{r^2}$$

and $\underline{u} = (0, v, w)$ is the equilibrium flow,

$$H = \underline{k} \cdot \underline{B}$$

$$B_*^2 = \rho a^2 + B^2$$

$$\beta = \frac{\rho a^2}{B_*^2} = \frac{a^2}{a^2 + b^2}$$

and $b^2 = \frac{B^2}{\rho}$ is the Alfvén speed,

$$p_*^o = p_o + \frac{1}{2} B_o^2 .$$

The matrix elements of Λ^o are

$$\lambda_{11}^o = -C[\frac{2m}{r^2} \sigma(B_\theta H - \rho\sigma v) + A\sigma' - \frac{1}{r} A\sigma] - \frac{1}{r}\rho^2\sigma^3 v^2 A + \frac{2}{r}\rho^2\sigma^4 B_\theta (Hv - B_\theta \sigma)$$

$$\lambda_{12}^o = \sigma^2(\rho^2\sigma^4 + CK^2)$$

$$\lambda_{21}^o = C[Ar(\frac{1}{r} p_*^{o\prime})' + A^2 - \frac{4}{r^2}(B_\theta H - p\sigma v)^2] - [\frac{1}{r} \rho v^2 A$$

$$- \frac{2}{r} \rho\sigma B_\theta (Hv - B_\theta \sigma)]^2$$

$$\lambda_{22}^o = -\lambda_{11}^o - AC\sigma' + \frac{1}{r} AC\sigma$$

System (A) is supplemented with two boundary conditions

(A1) $u = 0$ on the wall $r = R$

(A2) ru is regular near the origin $r = 0$.

Every system of the form (A) can be reduced to a single second order equation in any of the unknowns. e.g.

(A) $$(pu')' + qu = 0$$

where $p = \frac{rAC}{\lambda^o_{12}}$

$$q = \frac{r}{\sigma}\left[\frac{\det \Lambda^o}{\sigma AC \cdot \lambda^o_{12}} + \left(\frac{\lambda^o_{11}}{\lambda^o_{12}}\right)'\right]$$

Although equation (A) has a self adjoint form, it does not have this property. The coefficients p,q contain ω which may be complex, thus destroying the self adjointness with respect to the relevant scalar product $(u,v) = \int_0^R u\bar{v}dr$. It should be pointed out that every second order linear ODE can be brought into form (A) by multiplying with an appropriate factor, and that the self adjoint form has no meaning of its own. We choose it for reasons of convenience in handling the associated Green's function, and also in order to show the similarity to the equation derived by Hain and Lüst [4] for the static screw pinch. The two equations are in fact identical when we let the equilibrium flow vanish.

CHAPTER II: The Spectrum

II.1 Definitions

The term "spectrum" is properly used in the context of operators in a topological space - most commonly Hilbert space. The present formulation of the stability problem deals with the growth in time of solutions to the linearized Lundquist equation. We did not introduce a space to which the eight initial perturbations $\rho, p, \underline{u}, \underline{B}$ had to be restricted, and in fact there does not seem to be a "natural" space associated with this problem, with a norm that would for example provide a spectral expansion or have other advantages of that kind. The energy norm $\int \rho_0 |\underline{u}|^2 dV$ which is used in the static case, would limit consideration to perturbations of finite kinetic energy but would offer no mathematically exploitable structure to the problem at hand. The only clue we have is that the "right" norm, if there is any, must measure the growth of three perturbed quantities, since they all depend on $p_\star, u, u'$. It will be shown in chapter V that the Lagrangian formulation of the problem does offer an elegant way of turning the stability question into a non-linear eigenvalue problem in a Hilbert space. In the meantime, however, we restrict the discussion to a purely differential-equation context, although we continue to use the term "spectrum" for simplicity and after proper definition.

Our guide for the use of this term is the Laplace transform which was applied in the normal mode analysis. Every perturbed quantity $\xi(r,\omega)$ which is a solution of the non-homogeneous Laplace transformed equation, corresponds to a time dependent perturbation $\tilde{\xi}$, obtainable by the inverse transform

$$\tilde{\xi}(r,t) = \frac{1}{2\pi} \int_{\Gamma} e^{i\omega t} \xi(r,\omega) d\omega \quad ,$$

where Γ is a horizontal line in the complex ω plane lying below all the singularities of $\xi(r,\omega)$. The hyperbolicity of the problem guarantees that such a line Γ exists, i.e. that $\xi(r,\omega)$ is analytic in ω for Im ω sufficiently negative. We can now continue $\xi(r,\omega)$ analytically and move the line of integration upward. As we pass a singularity of ξ, being either an isolated singularity or a singularity along a curve, we have to add to the integral along Γ, an integral around the singularity we passed. This process yields a "spectral resolution" of the time dependent solution in the eigenmodes of the system of equations, and accordingly we call the isolated singularities of ξ - "eigenvalues" of the problem, and the other singularities - the "continuous spectrum". The rest of this chapter is devoted to the identification and investigation of the different parts of the spectrum associated with the screw pinch problem.

II.2 The Continuous Spectrum

In all subsequent discussions we consider the wave numbers m and k to be fixed. System (A') indicates the existence of a continuous spectrum which contains all points ω for which the system is singular at some point r in the range of the plasma. Although the spectrum is associatied with solutions of the non-homogeneous problem, it can be detected from the non-analyticity of solutions of the homogeneous problem simply because the Green's function of the problem is constructed by solutions of the homogeneous equation. There are three parts to the continuous spectrum:

i. The Alfvén continuum: $A = H^2 - \rho\sigma^2 = 0$

ii. The cusp continuum: $C \propto \beta H^2 - \rho\sigma^2 = 0$

iii. The Eulerian continuum: $\sigma = 0$.

It should be pointed out that although points where $\lambda_{12}^{o} = 0$ appear to be singularities of equation (A), they are in fact regular points, as seen from the equivalent system (A'). This apparent singularity results from the elimination of $p_\star$ as an unknown. If we eliminate u instead, we will get λ_{21}^{o} as an apparent singularity.

The first two continua are known in the static equilibrium case[5], and have there identical expressions, with σ replaced by ω. The last continuum reduces to the point $\omega=0$ when the flow vanishes, and is also a known singularity in the static case. The source of the names

assigned to the first two continua is their similarity to characteristic speeds of the Lundquist equation [3]. The Alfvén continuum is defined by an expression identical to the Alfvén wave propagation speed in the direction $\underline{k}$. The cusp continuum, $\underline{\text{if}}$ treated as a propagating signal, would yield propagation along a magnetic field line with a speed equal to that of the trailing cusp of the slow magnetosonic wave. It will be shown in chapter V that the Eulerian continuum is characteristic of the Eulerian description of the motion and is missing when Lagrangian formulation is used.

Because of the apparent similarity between continua and wave propagation speeds, it should be stressed that these two phenomena are not directly related, as pointed out by Grad [5]. Firstly, the fast and slow waves do not have counterparts among the continua. Secondly, the cusp continuum is not a signal speed. The resemblance can be explained by the formal mathematical procedures that lead to the determination of characteristic speeds and continua. In deriving the characteristics of a first order partial differential equation, we make the change $\frac{\partial u}{\partial x} \rightarrow in_x \delta u$ for every differential term, with n_x a constant. This looks much like normal mode analysis which proceeds similarly, except that r-derivatives are kept, and non-differentiated terms are not dropped. The continuum is determined by the vanishing coefficient of the r-derivative. Thus, for ω in

the continuum much of the difference between the two processes is eliminated because there are no r-differentiated terms, and hence the similarity.

Before concluding this part, we observe that the continuous spectrum consists only of real numbers ω and therefore does not cause exponential growth or decay of the perturbation. The question of exponential stability is thus decided by the point spectrum, and we proceed with its investigation.

II.3 Accumulating Eigenmodes

We first make the useful transformation

(2) $u = i\sigma\xi$

which uses ξ, the Lagrangian radial displacement, as the new unknown. The transformation fails when σ vanishes, i.e. when ω is in the Eulerian spectrum, but the point spectrum, which is the subject of this investigation, is not affected. System (A') now takes the form

$$(B') \qquad AC\left(\frac{\xi}{p_*}\right)' + \Lambda\left(\frac{\xi}{p_*}\right) = 0$$

where the matrix elements are

$$\lambda_{11} = -C\left[\frac{2m}{r^2}(B_\theta H - \rho\sigma v) - \frac{1}{r}A\right] + \frac{2}{r}\rho^2\sigma^3 B_\theta\,(Hv - B_\theta\sigma) - \frac{1}{r}\rho^2\sigma^2 v^2 A$$

$$\lambda_{12} = \rho^2\sigma^4 + CK^2$$

$$\lambda_{21} = C\left[Ar\left(\frac{1}{r}\,p_*^{o'}\right)' + A^2 - \frac{4}{r^2}(B_\theta H - p\sigma v)^2\right] - \left[\frac{1}{r}\rho v^2 A - \frac{2}{r}\rho\sigma B_\theta\,(Hv - B_\theta\sigma)\right]^2$$

$$\lambda_{22} = -\lambda_{11} + \frac{1}{r}AC$$

or equivalently

$$(B) \qquad (p\xi')' + q\xi = 0$$

$$p = \frac{rAC}{\lambda_{12}}$$

$$q = r\left[\frac{\det\Lambda}{AC\lambda_{12}} + \left(\frac{\lambda_{11}}{\lambda_{12}}\right)'\right]$$

The boundary conditions become

(B1) $\xi = 0$ at the outer wall $r = R$.

(B2) $r\xi$ is regular near the origin.

The transformation to the Lagrangian variable conceals the spectrum $\sigma = 0$ and makes equation (B) easier to handle. It does not however eliminate this continuum from the system. When solving the initial value problem we still have $\frac{1}{\sigma}$ terms in the non homogeneous side of (B). We also have such terms when expressing the other perturbed quantities ($\underline{B}$, etc.) in terms of $p_\star$, ξ and ξ'.

The point spectrum consists of all numbers ω not in the continuum, for which there exists a non trivial solution to equation (B), which satisfies the boundary conditions (B1-B2). Clearly, any non real contribution to the coefficients p,q of (B) is due to ω. By taking the complex conjugate of (B) we find that if ω is an eigenvalue, so is its complex conjugate $\bar{\omega}$ (with an eigenfunction $\bar{\xi}$).

Corollary: The flow is exponentially stable if and only if there are no non-real eigenvalues.

Rather than investigating the question of the existence or non-existence of complex eigenvalues, we tend to the simpler problem of whether or not there exists an accumulating sequence of unstable eigenvalues. The existence of such a sequence will be referred to as "local instability"

for reasons that will be made clear in chapter **III**. Our interest in local stability stems from the fact that in the steady screw pinch case local stability indicates global stability in current experimental ranges of the plasma parameters [6].

One way of finding the eigenvalues, is to start with a solution ξ of (B) which satisfies (B2) at $r = 0$, and in addition satisfies another arbitrary condition at the origin independent of ω. The eigenvalues are then the roots of the equation $\xi(R,\omega) = 0$, which is condition (B1). $\xi(R,\omega)$ is an analytic function of ω for ω not in the Alfvén or cusp continuum. The principle of isolated zeroes of analytic functions implies that the zeroes can only accumulate at a singularity of the function, namely the Alfvén and cusp continua, or possibly the point at infinity.

Claim: The point at infinity is not an accumulation point of complex eigenvalues.

Proof: If ω is an eigenvalue with an eigenfunction $\xi(r,\omega)$, we multiply equation (B) by $\bar{\xi}$ and integrate by parts. This yields

$$\int_0^R (p|\xi'|^2 - q|\xi|^2)dr = 0. \tag{3}$$

There is no contribution from the boundary because ξ satisfies (B1-B2). The coefficients p,q of (B) are

rational functions of ω and possess the following expansion near $\omega = \infty$:

$$p = p_o + p_2 \frac{1}{\omega^2} + O(\frac{1}{\omega^3})$$

(4)

$$q = q_o \omega^2 + O(\omega)$$

where p_o, p_2, q_o are positive functions for all $r \neq 0$ and are given by

$$p_o = rB_*^2$$

$$p_2 = \frac{r}{\rho} B_*^2 (K^2 B_*^2 - (\beta+1)H^2)$$

$$q_o = r\rho$$

(We add the assumption that ρ, a^2, B_*^2 do not vanish anywhere in the plasma). The imaginary part of (3) has the form

$$\int (\text{Im } p) |\xi'|^2 - (\text{Im } q) |\xi|^2 = 0 \quad . \tag{5}$$

For $|\omega|$ large enough but not real, Im q and -Im p have the same sign except when ω is near the imaginary axis. Therefore (5) can only be satisfied for such ω. To exclude this possibility we take the real part of (3).

$$\int (\text{Re } p) |\xi'|^2 - (\text{Re } q) |\xi|^2 = 0 \tag{6}$$

Re q is negative for ω large enough and near the imaginary axis, while Re p is positive and (6) is impossible.

The last proof excludes accumulation near infinity of complex eigenvalues only. It does not

exclude accumulation of real eigenvalues. Indeed, the asymptotic behavior of solutions of (B) for large ω is [7]

$$\xi = \cos[\omega \int^{r} (\frac{q_o}{p_o})^{\frac{1}{2}} dr][f(r)+O(\frac{1}{\omega})]$$

which verifies the existence of two sequences of real eigenvalues converging to $\pm\infty$.

In order to proceed with the identification of possible accumulation points, we make the important Assumption: All equilibrium profiles are analytic functions of r, in a neighborhood of the real interval [0,R].

From the preceding discussion we know that complex eigenvalues can accumulate only on the Alfvén or cusp continua. We now try to locate the accumulation points in the continuous spectrum. Let ω_o be in the Alfvén continuum, and let r_o be such that $A(r_o,\omega_o) = 0$. Let also $A'(r_o,\omega_o) \neq 0$ (where the prime denotes partial derivative with respect to r). From the implicit function theorem it follows that there exists an analytic function $r(\omega)$, defined in a neighborhood of ω_o, such that $A(r(\omega),\omega) = 0$ and $r(\omega_o) = r_o$. Moreover, $r(\omega)$ is only real for real ω(in the neighborhood of ω_o). This follows from the identity

$$\frac{dA}{d\omega} = 0 = \frac{dr}{d\omega} A' + \frac{\partial A}{\partial \omega} .$$

But $\frac{\partial A}{\partial \omega} = -2\rho\sigma$ which cannot vanish, otherwise $\sigma = H = 0$ and $A' = 0$ contrary to our assumption. We thus have $\frac{dr}{d\omega} \neq 0$, and since r is real for real ω, it cannot be real for ω non real.

In the complex r plane, equation (B) has a singularity at $r = r(\omega)$ and this is the only singularity of the equation in a fixed, small enough neighborhood of r_o, for ω close to ω_o. Let ω have a positive imaginary part, and let $r(\omega)$ have a negative imaginary part, say. Let $\xi(r,\omega)$ be an analytic solution (in ω) of (B) which satisfies (B2). The value $\xi(R,\omega)$ can be obtained by continuing ξ analytically along a path Γ in the upper half r plane (see fig. 1).

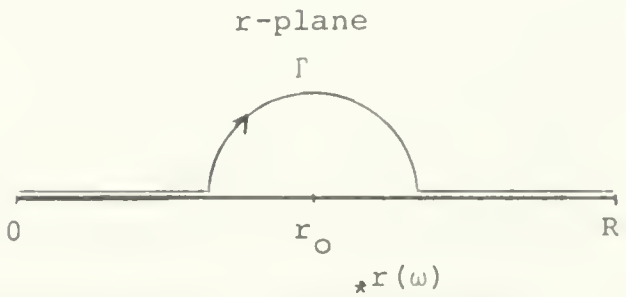

FIG. 1

When $\omega \to \omega_o$ equation (B) remains regular along Γ, hence $\xi(R,\omega)$ evaluated this way is analytic in ω and cannot have infinite number of zeroes near ω_o. A similar result is derived for ω in the lower half plane.

This process shows that the part of the continuous spectrum which satisfies the assumption above (of being a simple zero), is a cut in the domain of analyticity of $\xi(R,\omega)$. The function can be continued analytically through

the cut from above and below. A similar property was derived by Tataronis [8].

The previous argument yields similar results for ω_o in the cusp continuum. We are left with only two types of points as candidates for accumulation points:

i) A "tip" of the continuous spectrum, which is defined for the Alfvén part as a solution ω_o (with corresponding r_o) of the system of equations $A(r,\omega) = A'(r,\omega) = 0$, and a similar definition for the cusp continuum.

ii) The point of overlapping of the two continua. At this point $A = C = 0$, hence $H = \sigma = 0$. We can thus identify this point as follows: Let $H(r_o) = 0$ (if such a point exists), then $\omega_o = -\underline{k}\cdot\underline{u}(r_o)$. In this treatment we exclude the possibility $H'(r_o) = 0$.

The second case, although included in the first one, is singled out because of the different pattern of singularities of the differential equation near r_o. In case (i) there are two singularities near r_o in the complex r plane, while in case (ii) the equation has four singularities originating in pairs from the two continua. Case (ii) reduces in the static case to $H(r_o) = 0$, $\omega_o = 0$ which is known to be the only accumulation point of complex eigenvalues. It will be shown in Chapter IV that a similar situation occurs in the steady case. The following chapter is devoted to the development of the mathematical tools required for the analysis of our case.

CHAPTER III: An Asymptotic Expansion

III.1 The Overlapping of Continuous Spectra

We concluded chapter II by establishing that the point of overlap of the Alfvén and cusp continua is the one most likely to serve as an accumulation point of complex eigenvalues. The asymptotic method to be developed in this chapter is tailored for the investigation of this case. Although it is possible to develop the method after stating general assumptions, we prefer to use our special problem as an explanation for every property we make use of.

We make the change of variables

$$x = r - r_o \quad , \quad \varepsilon = \omega - \omega_o$$

where r_o and ω_o correspond to the overlapping point of the continua. Without loss of generality we assume that x ranges in the interval [-1,1]. We rewrite (B) as

$$(C) \qquad [p(x,\varepsilon)u']' + q(x,\varepsilon)u = 0 \quad , \quad -1 \leq x \leq 1$$

The proper boundary condition at x = -1 is a regularity condition like(B2). However, all we use is the analyticity of the solution with respect to the parameter. We therefore can assume for simplicity, that x = -1 is a

regular point of (C) and supplement the equation with the two boundary conditions

(C1) $$u(-1,\varepsilon) = 0$$

(C2) $$u(1,\varepsilon) = 0 \ .$$

we recall the leading coefficients of (B) $p = \frac{rAC}{\rho^2\sigma^4+CK^2}$. For $\omega = \omega_o(\varepsilon = 0)$ p vanishes with its first derivative at $r = r_o(x = 0)$, thus $p = O(x^2)$ near the origin. It is also seen that if we let

(7) $$z = \frac{x}{\varepsilon}$$

then $\frac{1}{\varepsilon^2}\, p(\varepsilon z,\varepsilon)$ converges as $\varepsilon\to 0$ and z held fixed to a function of z with a second order pole at infinity. Likewise we find that for $\varepsilon = 0$, $q(x,o)$ approaches a constant as $x\to 0$. Also $q(\varepsilon z,\varepsilon)$ tends to the same constant when first $\varepsilon\to 0$ and then $z\to\infty$.

Employing these properties of p,q we use (7) to rewrite (C) as an equation in z

(D) $$(\tilde{p}(z,\varepsilon)\tilde{u}') + \tilde{q}(z,\varepsilon)\tilde{u} = 0$$

$$\tilde{p}(z,\varepsilon) = \frac{1}{\varepsilon^2}\, p(\varepsilon z,\varepsilon)$$

$$\tilde{q}(z,\varepsilon) = q(\varepsilon z,\varepsilon)$$

where the prime denotes differentiation with respect to z.

To find the "eigenvalues" ε of (C) we use a boundary layer analysis similar to the one used by Pao [9] in his investigation of the static screw-pinch equilibrium, but with a substantial modification required by the possibility that the eigenvalues are not resticted to the imaginary axis. In most cases asymptotic methods are believed to yield correct results but proofs are not available. This is not the case with the process we are about to describe. The proof is omitted from this chapter for the sake of clarity, but will be presented in much detail in Appendix A.

We divide the interval [-1,1] into three regions:

(i) The outer region on the right, in which $x \gtrsim |\varepsilon|$.

(ii) The outer region on the left, $x \lesssim -|\varepsilon|$.

(iii) The inner region $|x| \lesssim |\varepsilon|$.

In the outer regions ε is negligible with respect to x, while in the inner region we neglect ε with respect to z. To leading order in ε, an eigenfunction of (C) is taken to satisfy (C) and (C1-C2) in the outer regions, when ε is set to zero, and to satisfy equation (D) in the inner region again with $\varepsilon=0$. The matching between the different regions is done in the following way. We note that after setting $\varepsilon=0$ in (C) and (D) the two equations have singularities of the first kind (regular singularities) at $x=0$ and $z=\infty$ respectively. Moreover,

the indicial equations that determine the behavior of solutions near the regular singularity, are identical in both cases. They are of the form

$$\nu(\nu+1) + a = 0 \tag{8}$$

where the real constant a is easily derived from the coefficients of (C) after setting $\varepsilon = 0$. (8) has the two solutions ν and $-(\nu+1)$, with ν chosen so that $\text{Re } \nu \geq -\frac{1}{2}$ and $\text{Im } \nu > 0$ when $\text{Re } \nu = -\frac{1}{2}$. We now require that the outer solution with behavior $x^{-(\nu+1)}$ near $x = 0$, be matched with $\varepsilon^{-(\nu+1)}$ times the inner solution with behavior $z^{-(\nu+1)}$ near $z = \infty$, and that x^{ν} be matched with $\varepsilon^{\nu}[z^{\nu}+cz^{-(\nu+1)}]$ where c is some constant determined in Appendix A. c is zero in the important case of $\text{Re } \nu = -\frac{1}{2}$ (ν complex). In this last case, the amplitude of the eigenfunction in the inner region is $\varepsilon^{-\frac{1}{2}}$ times larger than the amplitude in the outer region, which explains the name "local mode".

III.2 The Inner Region Equation

The coefficients of (D) (after setting $\varepsilon=0$) can be derived from (C) by substituting

$$H \to hz$$

(9)

$$\sigma \to \alpha z+1$$

where $h = H'(x=0)$, $\alpha = (\underline{k}\cdot\underline{u})'(x=0)$.
All other functions are taken to equal their constant value at $x = 0$. We find

$$\tilde{p}(z) = \frac{r}{k^2} A \tag{10}$$

$$\tilde{q}(z) = \frac{4k^2}{rK^2A} (B_\theta H-\rho\sigma v)^2 + \frac{1}{rAC}[\rho v^2 A-2\rho\sigma B_\theta (Hv-B_\theta\sigma)]^2$$

$$-r^2(\frac{1}{r} p_\star^{o\prime})' - \frac{2m}{rK^2}(B_\theta h-\rho\alpha v)$$

where H and σ are now defined by (9), $r = r_o$ and as before

$$A = H^2-\rho\sigma^2 \quad , \quad C = B_\star^2(\beta H^2-\rho\sigma^2).$$

The inner region equation now has coefficients which are rational functions of z, and its finite points of singularity are determined by the vanishing of A and C. There

are four such points

$$z_{1,2} = \frac{1}{-\alpha \pm \frac{h}{\sqrt{\rho}}} \quad , \quad z_{3,4} = \frac{1}{-\alpha \pm \frac{\sqrt{\beta}\, h}{\sqrt{\rho}}} \tag{11}$$

The point $z = \infty$ is another regular singularity of the equation. We can choose two independent solutions of the equation such that

$$\begin{aligned} v_1 &= z^{\nu} f_1(z) \\ v_2 &= z^{-(\nu+1)} f_2(z) \end{aligned} \tag{12}$$

where ν is given by (8) and f_1, f_2 are analytic functions of z in a full neighborhood of infinity, and satisfying $f_1(\infty) = f_2(\infty) = 1$. As seen from (12), the two solutions are multivalued functions and we must introduce a cut in the z plane that extends to infinity. We choose to cut the plane along the negative real axis, and to use $-\pi < \arg z \leq \pi$ in (12).

The transformation (7) maps the x interval [-1,1] onto a line segment passing through the origin, in the complex z planes. (see fig. 2). Every solution of (D) along

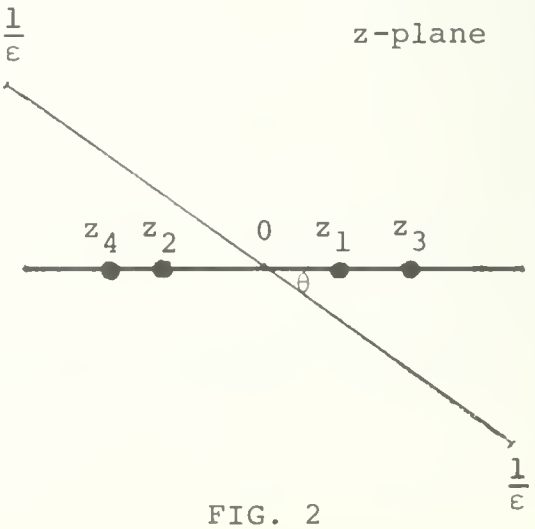

FIG. 2

the line, can be continued analytically to the complex plane, but cuts have to be introduced because of the presence of finite singularities. We cut the plane along the real axis from each singularity to infinity, leaving a neighborhood of the origin in the domain of regularity of the solution.

We restrict consideration to ε in the upper half plane, so that

$$0 < \theta = \arg \varepsilon < \pi \ . \tag{13}$$

Transformation (7) then maps $x>0$ into the lower half z plane, while $x < 0$ is mapped into the upper plane. The matching of innner region solutions with outer region solutions on the right requires, that we know their behavior near infinity in the lower z plane, while the matching on the left depends on the behavior of solutions of (D) in the upper half plane. In order to construct a solution of (C) in the full interval $[-1,1]$, we need to know how solutions of (D) change their behavior when continued from the upper half plane to the lower one in the cut plane of figure 2. It is sufficient to know how v_1, v_2 of (12) change their behavior. We describe it symbolically as

$$\begin{aligned} z^{\nu} &\to \alpha z^{\nu} + \beta z^{-(\nu+1)} \\ z^{-(\nu+1)} &\to \gamma z^{\nu} + \delta z^{-(\nu+1)} \end{aligned} \tag{14}$$

where α, β, γ,δ are the "connection coefficients" (not to be confused with the same symbols used for different purposes elsewhere in the work), and the arrow points to analytic continuation from the upper to the lower plane. z^{ν}, $z^{-(\nu+1)}$ stand for v_1, v_2.

The connection relations (14) do not change if v_1, v_2 are continued in the positive direction around all the singularities of (D) which are in the right half plane (see fig. 3). This is because the cut in the domain of regularity of the two functions is along the negative real line and not the positive one. Clearly, if (D) has no singularities in the right half plane, then $\alpha = \delta = 1$, $\beta = \gamma = 0$.

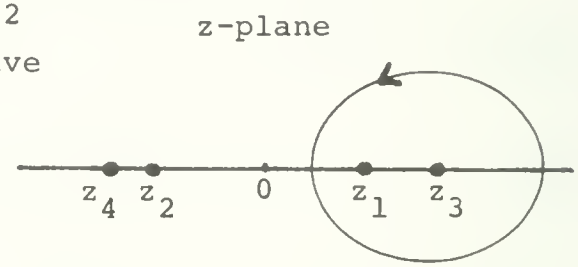

FIG. 3

III.3 Matching the Different Regions

We begin with a solution u of (C) (with $\varepsilon = 0$) in the outer region on the right, which satisfies (C2). Near the origin $u \sim A(x^{\nu}+rx^{-(\nu+1)})$ with r a constant and independent on ε. x^{ν}, $x^{-(\nu+1)}$ stand for the two solutions of (C) with the corresponding behavior. u is matched with a solution v of (D) with the behavior $v \sim A[\varepsilon^{\nu}(z^{\nu}+cz^{-(\nu+1)})+ r\varepsilon^{-(\nu+1)}z^{-(\nu+1)}]$ in the lower half plane. v is continued into the upper plane and has to be matched with a solution w of (C) which satisfies (C1) and thus behaves like $w \sim a(x^{\nu}+\ell x^{-(\nu+1)})$ for $x < 0$ ($\arg x = \pi$), with the constant ℓ independent on ε. Using (14), matching is possible if and only if

$$\alpha\varepsilon^{\nu} + \gamma(c\varepsilon^{\nu}+\ell\varepsilon^{-(\nu+1)}) = \frac{A}{a}\varepsilon^{\nu}$$

(15)

$$\beta\varepsilon^{\nu} + \delta(c\varepsilon^{\nu}+\ell\varepsilon^{-(\nu+1)}) = \frac{A}{a}(c\varepsilon^{\nu} + r\varepsilon^{-(\nu+1)})$$

Let

(16) $$\tau = \varepsilon^{2\nu+1}$$

then (15) can be written as

(17) $$\alpha + \gamma(c + \frac{\ell}{\tau}) = \frac{A}{a}$$

$$\beta\tau + \gamma(c\tau + \ell) = \frac{A}{a}(c\tau + r)$$

By its definition Re $\nu \geq -\frac{1}{2}$. If Re $\nu > -\frac{1}{2}$ then $\tau \to 0$ as $\varepsilon \to 0$. System (17) does not have a sequence of solutions τ_n converging to zero. We conclude that a sequence of complex eigenvalues $\varepsilon_n \to 0$ can exist only if Re $\nu = -\frac{1}{2}$. From (8) we have

$$\nu = \frac{1}{2}[-1+\sqrt{1-4a}]$$

and a necessary condition for the existence of a sequence of complex eigenvalues near $\varepsilon = 0$ is

$$a > \frac{1}{4} \quad . \tag{18}$$

In the static equilibrium case, condition (18) is the Suydam condition [10] and is known to be a necessary and sufficient condition for local instability [11].

We now assume that relation (18) holds, and define

$$\mu = \sqrt{4a-1} \tag{19}$$

$$\theta = \arg \varepsilon \tag{20}$$

then $\tau = \varepsilon^{i\mu} = \exp(-\mu\theta + i\mu \log|\varepsilon|)$. Every solution τ of (17) corresponds to a sequence of numbers ε

$$\varepsilon_n = \varepsilon_o \exp\left(-\frac{2\pi n}{\mu}\right) \quad , \quad \arg \varepsilon_n = \arg \varepsilon_o = \theta \tag{21}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

with θ determined by

$$|\tau| = e^{-\mu\theta} \tag{22}$$

As $n \to \infty$, $\varepsilon_n \to 0$ which is consistent with the assumption underlying the asymptotic expansion. From (21) we see that asymptotically, the sequence of eigenvalues is on a ray θ = constant, with a spacing of $\exp(-\frac{2\pi}{\mu})$ between consecutive members. In the static equilibrium case it is known that ε^2 is real, thus $\theta = \frac{\pi}{2}$.

Equation (17) is equivalent to the quadratic equation

$$\beta\tau^2+(\ell\delta - r\alpha)\tau - r\ell\gamma = 0 \tag{23}$$

where we used the fact that c = 0 in (17) when $\mathrm{Re}\nu = -\frac{1}{2}$. It is seen that in principle two sequences of complex eigenvalues are possible.

The following property of r and ℓ is straightforward, but of great importance. We claim

$$|r| = 1 \quad , \quad |\ell| = e^{-\mu\pi} \tag{24}$$

Proof: Equation (C) has real coefficients, and so are the real boundary conditions. Hence, in the outer region on the right, the eigenfunction is a multiple of the real

solution with behavior

$$x^{\nu} + rx^{-(\nu+1)} = x^{-\frac{1}{2}}(x^{i\mu} + rx^{-i\mu}) \quad .$$

Since we chose arg x = 0 for x > 0, the reality of this solution requires that $|r| = 1$. The proof for ℓ is identical except for the change in the argument of x; arg x = π for x < 0.

In view of (24) we define

$$r = e^{i\rho} \quad , \quad \ell = e^{-\mu\pi+i\lambda} \tag{25}$$

with ρ, λ real. We now prove a few helpful properties of the connection coefficients.

Property 1:

$$\det \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1$$

Proof: Every two solutions ξ_1, ξ_2 of (D) (which is of a self adjoint form) satisfy $\tilde{p}(z)[\xi_1, \xi_2] =$ constant, where $[\xi_1, \xi_2]$ is the Wronskian of the two solutions. As we continue the solutions around the singularities in the right half plane, we get for the functions (12)

$$\tilde{p}(z)\ [v_1, v_2] = \tilde{p}[\alpha v_1 + \beta v_2,\ \gamma v_1 + \delta v_2] = (\alpha\delta - \beta\gamma)\ \tilde{p}[v_1, v_2].$$

Property 2: All connection matrices associated with the same path of continuation, are similar.

proof: Let $\underline{u}$, $\underline{v}$ denote two pairs of independent solutions of (D) with the connection relations

$$\underline{u} \to A\underline{u} \quad , \quad \underline{v} \to B\underline{v} \ .$$

We can express the solutions $\underline{u}$ as a linear combination of $\underline{v}$: $\underline{u} = S\underline{v}$ with a constant matrix S.

$$\underline{u} = S\underline{v} \to SB\underline{v} = A\underline{u}$$

hence $B = S^{-1}AS$.

property 3: α and δ are real, $\gamma = -\bar{\beta}$

Proof: Equation (D) has real analytic coefficients. By the Schwartz reflection principle we have

$$\tilde{p}(\bar{z}) = \overline{\tilde{p}(z)} \quad , \quad \tilde{q}(\bar{z}) = \overline{\tilde{q}(z)}$$

Hence, every solution v(z) gives rise to another solution $\bar{v}(z)$ which is its reflection in the real axis, $\bar{v}(z) = \overline{v(\bar{z})}$. For z in the upper half plane we find

$$\bar{v}_1(z) \sim \bar{\alpha} z^{-\frac{1}{2} - i\frac{\mu}{2}} + \bar{\beta} z^{-\frac{1}{2} + i\frac{\mu}{2}}$$

thus $\bar{v}_1 = \bar{\beta}v_1 + \bar{\alpha}v_2$. Similarly for z in the lower half plane we get $\bar{v}_1 = -\gamma v_1 + \bar{\alpha}v_2$. Comparison of the two results yields the desired relation. The proof for δ is similar when we consider $\bar{v}_2$. It is noted that in the proof we used the range $-\pi < \arg z \leq \pi$.

property 4: If in equation (D), the coefficients are analytic functions of z^2 rather than of z, we have $\alpha + \delta e^{-\mu\pi} = 0$.

Proof: The coefficients of (D) are real on the imaginary axis. Thus, to every solution v corresponds another solution $\tilde{v}$ which is the reflection of the former in the imaginary axis, i.e. $\tilde{v}(z) = \overline{v(-\bar{z})}$. We carry now a procedure similar to the one used before. We note that for z in the upper plane $-\bar{z} = e^{\pi i}\bar{z}$, while for z in the lower plane $-\bar{z} = e^{-\pi i}\bar{z}$. The difference between the two cases is determined by the choice of arg z. Thus, we arrive at the relations

$$\bar{\beta}(\alpha + \delta e^{-\mu\pi}) = 0 \quad , \quad |\beta|^2 - \alpha^2 e^{\mu\pi} = 1$$

from which it follows that $\beta \neq 0$ and $\alpha + \delta e^{-\mu\pi} = 0$.

From equation (23), the two solutions of the quadratic equation satisfy the relation

$$\tau_1\tau_2 = -r\ell\,\frac{\gamma}{\beta}$$

In view of (24) and property (3), we have $|\tau_1\tau_2| = e^{-\mu\pi}$. Let $|\tau_i| = e^{-\mu\theta_i}$ (i=1,2), then $\theta_2 = \pi-\theta_1$ and we have shown the

Result: If $\varepsilon=0$ is an accumulation point of complex eigenvalues, then they converge (asymptotically) along two rays, which are the reflection of each other in the imaginary axis (see figure 4). (The rays, not the eigenvalues, are reflected). The eigenvalues in the lower half plane are the complex conjugate of those in the upper half plane.

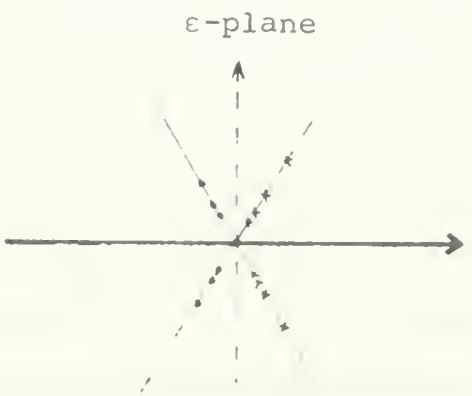

FIG. 4

It was assumed all along that $0<\theta<\pi$. As was pointed out before, this range of arg ε was used in determining the connection coefficients as well as in deriving some properties of them. Thus, only solutions of (23) which correspond to this range of θ, give rise to local instability. In order to find when this happens we rewrite (23) as

$$\tau + r\ell\,\frac{\bar{\beta}}{\beta}\cdot\frac{1}{\tau} = \frac{r\mu-2\delta}{\beta}$$

Using (25) and defining

$$\tau = s \exp \frac{i}{2}(\rho+\lambda-2b) \tag{26}$$

$$\beta = |\beta| e^{ib} \quad , \quad \psi = \frac{\rho-\lambda}{2}$$

we have

$$s + \frac{e^{-\mu\pi}}{s} = \frac{1}{|\beta|}\left[\alpha e^{i\psi} - e^{-\mu\pi}\delta e^{-i\psi}\right] \tag{27}$$

The angle of convergence θ can be determined by $|s| = |\tau| = e^{-\mu\theta}$. Let

$$F(s) = s + \frac{e^{-\mu\pi}}{s}$$

$$G(\psi) = \frac{1}{|\beta|}\left[\alpha e^{i\psi} - e^{-\mu\pi}\delta\ e^{-i\psi}\right] \tag{28}$$

The mapping F(s) transforms the circles in the s plane that are centered at the origin, into a family of confocal ellipses in the F plane with foci at $\pm 2 \exp(-\frac{\mu\pi}{2})$. In particular, the circle $|s| = e^{-\mu\pi/2}$ maps onto the line segment (degenerate ellipse) between the two foci. The transformation $G(\psi)$ maps the real line onto a horizontal or vertical ellipse (to be denoted by α) with major axes $\frac{1}{|\beta|}\ |\ \alpha \mp e^{-\mu\pi}\delta|$. This ellipse is not a member of the

former confocal family. Indeed, belonging to this family requires that $\alpha\delta+|\beta|^2 = 0$, a contradiction. We arrive at the

Result: As ψ varies, the solutions of (27) change their absolute value. Thus the angle of accumulation of complex eigenvalues depends on the global behavior of the equilibrium profiles and not only on their value at $x = 0$. In particular, the angle will change if we change the position of the outer wall.

This situation differs from the static equilibrium case, where the self adjointness of the problem provides that ε^2 is real and $\theta = \frac{\pi}{2}$. A derivation of this result based on our formulation is obtained as follows. When the flow vanishes, the coefficients of (D) depend on z^2 only. From property 4 we get $\alpha+\delta e^{-\mu\pi} = 0$, and the α-ellipse degenerates to a horizontal line segment. Equation (27) then yields $|s| = e^{-\frac{\mu\pi}{2}}$ regardless of ψ.

Proceeding with the investigation of (27), we require the solutions to satisfy $e^{-\mu\pi}<|s|<1$, corresponding to $\pi>\theta>0$. The point $G(\psi)$ must then be in the interior of the ellipse corresponding to $|s| = 1$ (and to $|s| = e^{-\mu\pi}$), which is characterized by its major axes $1 \pm e^{-\mu\pi}$. We denote this ellipse by μ. Three cases seem possible (see figure 5).

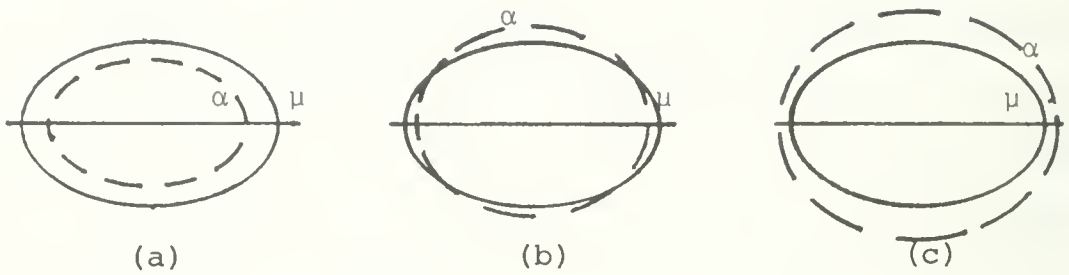

FIG. 5

case (a): α is in the interior of μ, which yields eigenvalues for all ψ.

case (b): α intersects μ. There exist eigenvalues for some ψ but not for all values.

case (c): α is in the exterior of μ. There are no eigenvalues near $\varepsilon = 0$.

In view of properties 1 and 3, the connection coefficients satisfy the relation

$$\alpha\delta + |\beta|^2 = 1 \tag{29}$$

which we use to express $|\beta|$ in terms of α and δ. The three cases can now be characterized in the following way. Let

$$Q(\alpha,\delta) = \alpha^2 e^{\mu\pi} + 2\cosh(\mu\pi)\alpha\delta + \delta^2 e^{-\mu\pi} \tag{30}$$

$$\begin{aligned} &\text{case (a) corresponds to } Q(\alpha,\delta) < 4\sinh^2\frac{\mu\pi}{2} \\ &\text{case (b): } 4\sinh^2\frac{\mu\pi}{2} < Q(\alpha,\delta) < 4\cosh^2\frac{\mu\pi}{2} \\ &\text{case (c): } Q(\alpha,\delta) > 4\cosh^2\frac{\mu\pi}{2}\,. \end{aligned} \tag{31}$$

The derivation of these inequalities simply uses comparison of the major axes of the two ellipses α,μ.

A further simplification in the characterization of the three cases (31) is required in order to enable us to determine which one of them holds, while bearing in mind that the connection coefficients α,δ are not known explicitly. We recall property 2 which states that all connection matrices associated with a path of analytic continuation are similar. In particular, their trace is independent of the pair of independent solutions we continue. Let tr_1 , tr_2 be the traces of the connection matrices associated with the paths Γ_1, Γ_2 winding around the singularities of (D) in the right and left half planes (see figure 6). We use the two solutions (12) to obtain the following relations

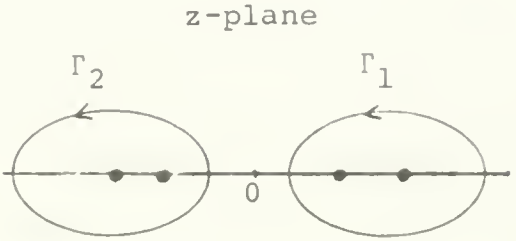

FIG. 6

(32)
$$\alpha+\delta = tr_1$$

$$\alpha e^{\mu\pi} + \delta e^{-\mu\pi} = - tr_2$$

from which α,δ can be determined in terms of tr_1, tr_2 . The factors in the second equation of (32) arise from the dependence on arg z in the definition of (12), which changes the connection relation of v_1 say, to

$$z^{\nu} \to \alpha(ze^{-2\pi i})^{\nu} + \beta(ze^{-2\pi i})^{-(\nu+1)}$$

when v_1 is continued along Γ_2 in the negative direction. The direction of continuation however, is immaterial for our purposes. The traces of a matrix and its inverse are equal if the determinant is unity (see property 1). We observe that by (32) we have

$$Q(\alpha,\delta) = -tr_1 \cdot tr_2 \quad . \tag{33}$$

If in particular the two traces (which by (32) are real) have equal signs, then only case (a) of (31) holds and eigenvalues are possible regardless of the global behavior of the plasma. We will pursue this point in chapter IV and show that this is indeed the situation for many plasma profiles.

Analytic determination of the two traces generally requires that we know explicitly two indpendent solutions of (D) (e.g. in the form of an integral representation).

However, if one of the half planes contains only one regular singularity, the associated trace can be determined by the characteristic exponents of solutions near the singularity. For the indicial exponents λ_1, λ_2, the trace is $e^{2\pi i \lambda_1} + e^{2\pi i \lambda_2}$. If a numerical determination of the connection coefficients is sought, relation (32) offers the great benefit of avoiding the necessity to compute to infinity, especially when the solutions of (D) are oscillatory.

We now proceed with proving the asymptotic nature of the matching procedure described in this chapter, after which we will resume the investigation of the screw pinch steady equilibrium.

Appendix A

The asymptotic expansion used in chapter III, will be carried out rigorously. We first describe the behavior of solutions of (C) in the outer and inner regions associated with the singularity at $x = 0$, $\varepsilon = 0$. Equation (C) has the form

$$(A1) \qquad (p(x,\varepsilon)u')' + q(x,\varepsilon)u = 0, \quad -1 \leq x \leq 1 .$$

Let $u(x,\varepsilon)$ be a solution of (A1) which behaves like x^{ν} near the origin, when $\varepsilon = 0$. We require that u satisfies two initial conditions at $x = -1$ independently of ε, such that it has the desired behavior. Let $\phi(x) = u(x,0)$ and define $v(x,\varepsilon)$ such that

$$(A2) \qquad u = \phi v$$

v then satisfies the equation

$$(A3) \qquad (p\phi^2 v')' = [\Delta q\phi^2 + \phi(\Delta p\phi')']v$$

with initial conditions

$$(A4) \qquad v(-1,\varepsilon) = 1 \quad , \quad v'(-1,\varepsilon) = 0$$

where

(A5) $\Delta q = q_o - q \quad , \quad \Delta p = p_o - p$

with $q_o = q(x,\varepsilon=0)$ and $p_o = p(x,\varepsilon=0)$.

We assume that ϕ does not have any zeroes in $-1 \leq x < 0$, otherwise we impose initial conditions for u and ϕ at $-1 < x_o < 0$. The right hand side of (A3) can be viewed as an inhomogeneous term, and it is seen that the homogeneous equation has the two independent solutions $1, \int^{x} \frac{1}{p\phi^2}$. v therefore, satisfies the integral equation

$$(A6)\quad v(x,\varepsilon) = 1 + \int_{-1}^{x} K(x,\xi)[\Delta q\phi^2 + \phi(\Delta p\phi')']v(\xi)d\xi = 1 + \hat{K}v$$

where

$$(A7)\quad K(x,\xi) = \int^{x} \frac{dt}{p\phi^2(x)}$$

We recall that $p = \frac{rAC}{\rho^2\sigma^4 + CK^2}$. (See the definitions in section I.2).

For ε and x small, A has two zeroes, at $x_{1,2} = h_{1,2}(\varepsilon)$ with h_i (i = 1,2) analytic functions of ε and $h_i(0) = 0$.

We can thus write p as

$$p = f(x,\varepsilon)\ (x-h_1(\varepsilon))(x-h_2(\varepsilon)).$$

We restrict our treatment to $|x|>N|\varepsilon|$, $|\varepsilon|<\delta$ with N large enough such that the four singularities near $x = 0$ are in the interior of $|x| = N|\varepsilon|$ for $|\varepsilon|<\delta$. Despite the denominator of p, $f(x,\varepsilon)$ can be uniformly bounded $0<c_1<|f|<c_2$ for $N|\varepsilon|<|x|<1$, $|\varepsilon|<\delta$.

The integral equation (A6) is solved by the Picard iterations method. We define $v_{n+1} = v_o + \hat{K}\, v_n$, $v_o = 1$. The integral in $\hat{K}$ is taken along a path composed of γ_1-the real interval $-1\leq\xi\leq-|x|$, and γ_2- the circular arc $|x|$ = const, from $-|x|$ to x in the negative direction (see fig. A1). Proceeding in the standard way, we look for a bound of $\hat{K}$ which will guarantee the convergence of the iteration scheme. In this appendix we use the notation

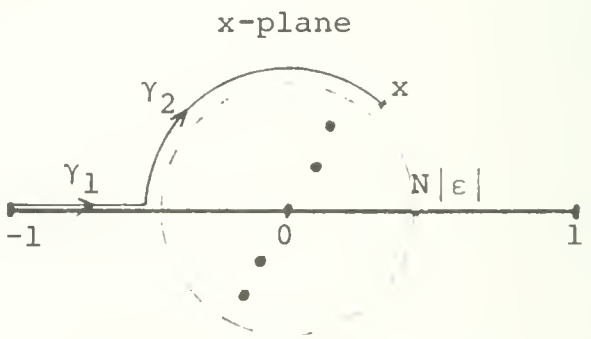

FIG. A1

$$x = \varepsilon z \quad , \ \xi = \varepsilon\zeta \quad , \ t = \varepsilon\tau \quad .$$

$$K(x,\xi) \equiv K(z,\zeta) = \int_{\xi}^{x} \frac{dt}{p\phi^2} = \frac{1}{\varepsilon^{2\nu+1}} \int_{\zeta}^{z} \frac{d\tau}{fg\ \tau^{2\nu}\ (\tau-\frac{h_1}{\varepsilon})\ (\tau-\frac{h_2}{\varepsilon})}$$

where $\phi = x^{\nu}g(x)$, and g is bounded away from zero. For $|\tau|>N$, the quadratic in the denominator is close to τ^2 (if N is so chosen), and we get (for ν real)

$$|K(z,\zeta)| \leq \frac{\text{const}}{|\varepsilon|^{2\nu+1}} \left[\int_{|z|}^{|\zeta|} \frac{d|\tau|}{|\tau|^{2\nu+2}} + \frac{2\pi}{|z|^{2\nu+1}} \right]$$

The last term is the contribution of the circular arc. We have

$$(A8) \quad |K(z,\zeta)| \leq \frac{1}{|\varepsilon|^{2\nu+1}} \left[\frac{c_1}{|z|^{2\nu+1}} + \frac{c_2}{|\zeta|^{2\nu+1}} \right] .$$

From their definition

$$\Delta q = O(\frac{1}{z})$$

$$\Delta p = O(\varepsilon x)$$

The two kernel terms $\phi^2\Delta q$ and $\phi(\Delta p\phi')'$ are of similar order when we fix z and let ε be small, and it is sufficient to treat one of them.

$$\left| \int_{\gamma_1} K(x,\xi)\Delta q\phi^2(\xi)d\xi \right| \leq \text{const} \int_{|z|}^{1/|\varepsilon|} \left[\frac{c_1}{|z|^{2\nu+1}} + \frac{c_2}{|\zeta|^{2\nu+1}}\right] |\zeta|^{2\nu-1} d|\zeta|$$

$$\leq \frac{A}{|z|} + B|\varepsilon| + C|\varepsilon| \, |\varepsilon z|^{-(2\nu+1)} \text{ with constants A,B,C.}$$

The integral along γ_2 has a bound of $\frac{A}{|z|}$, and can be incorporated in the previous result.

Let $$F(x,\varepsilon) = \int_{-1}^{x} |K(x,\xi)[\Delta q\phi^2 + \phi(\Delta p\phi')']d\xi|$$

then $|v_{n+1}-v_n| \leq \frac{F^{n+1}}{(n+1)}$, as can be seen by induction. We write $v_n = v_o+v_1-v_o+\ldots+v_n-v_{n-1}$. The sequence converges uniformly in a compact set of z. Let $v_n \to v$, then v satisfies the integral equation (A6), hence also the differential equation (A3), and we have

$$\text{(A9)} \qquad |v-1| \leq e^F-1 \quad , \quad F \leq \frac{A}{|z|} + B|\varepsilon| + c|\varepsilon|\,|\varepsilon z|^{-(2\nu+1)}$$

if $-(2\nu+1)\geq 0$, i.e.

$$\text{(A10)} \qquad \nu \leq -\frac{1}{2}$$

(which we now assume, although it is contrary to previous notation) then for fixed z, F remains bounded as $\varepsilon \to 0$.

We want to show that as $\varepsilon \to 0$ and z kept fixed, $v(z,\varepsilon)$ converges to a function $v_1(z)$ which is the solution of (A3) with $\varepsilon = 0$. ((A3) has to be multiplied by $\varepsilon^{-2\nu}$ in order for the coefficients to converge to non trivial functions of z). From (A6) we get

$$\frac{dv}{dx} = \frac{1}{p\phi^2(x)} \int_{-1}^{1} [\Delta q\phi^2 + \phi(\Delta p\phi')']v(\xi)d\xi$$

Hence

$$\left|\frac{dv}{dz}\right| \leq \frac{\text{const}}{|z|^{2\nu+2}} \left| \int_z^{-1/\varepsilon} \zeta^{2\nu-1} v(\zeta)\, d\zeta \right| \leq \text{const} \frac{e^{\left|\frac{A}{z}\right|}}{|z|^{2\nu+2}} \left[|z|^{2\nu} + c_1 |\varepsilon|^{-2\nu} \right]$$

which proves that $\frac{dv}{dz}$ remains bounded as $\varepsilon \to 0$.
By the Arzela-Ascoli Theorem there exists a subsequence $v(z,\varepsilon_n)$ converging uniformly to $v_1(z)$ as $\varepsilon_n \to 0$. Because of the uniform convergence, the limit function v_1 satisfies (A3) when z held fixed and $\varepsilon = 0$. Equation (A3) with $\varepsilon = 0$ has two independent solutions which behave like 1 and $z^{-(2\nu+1)}$ near $z = \infty$, of which the latter is the larger one by (A10). (A9) shows that v_1 behaves like 1 near infinity and thus is exactly the smaller solution. Since we can identify the limit function, it follows that $v(z,\varepsilon) \to v_1(z)$ for any sequence of ε. (Otherwise, there exists a sequence $v(z,\varepsilon_k)$ which stays away from $v_1(z)$, but a subsequence of it must converge to v_1).

From the definition $u = \phi v$ we find that as $\varepsilon \to 0$, $\varepsilon^{-\nu} u \to z^{\nu} v_1(z) \sim z^{\nu}$ where u is the large solution near $x = 0 (\nu \leq -\frac{1}{2})$. In the important case of Re $\nu = -\frac{1}{2}$ and ν complex, the proof follows throught for the two independent solutions.

The small solution $u \sim x^{-(\nu+1)}$ requires a similar treatment. We use the same iterative scheme, but this

time with

$$(A10) \qquad v_o = -(2\nu+1)\int_{N\varepsilon}^{x} \frac{dt}{p\phi^2} \times \varepsilon^{2\nu+1}$$

with a path of integration consisting of a circular arc and a ray, (see fig. A2). For $\varepsilon = 0, \varepsilon^{-(2\nu+1)} v_o \sim x^{-(2\nu+1)}$ near the origin and in fact, it converges to this function as $\varepsilon \to 0$ and x kept fixed. (Dominated Convergence Theorem). Carrying out the iteration process we find, that the limit function v exists and satisfies

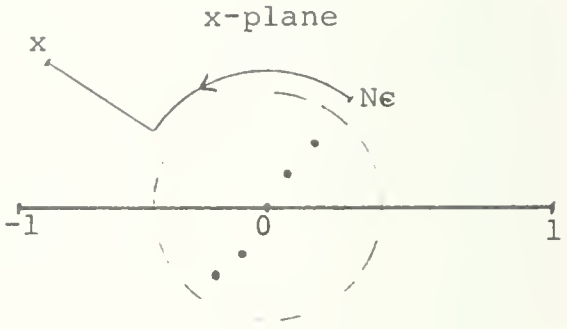

FIG. A2

$$(A11) \qquad |v-v_o| \leq (e^F-1)|v_o|$$

For fixed z

$$v_o = -(2\nu+1)\int_{N}^{z} \frac{d\tau}{fg\tau^{2\nu}(\tau-\frac{h_1}{\varepsilon})(\tau-\frac{h_2}{\varepsilon})}$$

and as $\varepsilon \to 0$ it converges to a function of z with a behavior of $z^{-(2\nu+1)}$ near infinity. Again, using the Arzela Ascoli theorem, there exists a sequence $\varepsilon_n \to 0$ such that $v(z,\varepsilon_n) \to v_2(z)$. By (A11), $v_2(z) \sim z^{-(2\nu+1)}$

near $z = \infty$ and therefore v_2 is the sum of this solution of (A3) (with $\varepsilon = 0$) and cv_1 with a constant c, and v_1 as defined before. The constant c is determined by the requirement that $v_2(z=N) = 0$, which follows from (A11) and $v_o(z=N) = 0$. The unique identification of the limit function enables us to conclude that $v(z,\varepsilon) \to v_2(z)$ for any sequence $\varepsilon \to 0$.

It is important to note that the iteration procedures used before, also converge when x is considered the argument and ε is small. As $\varepsilon \to 0$, the two limit functions converge to 1 and to $a(x) \cdot x^{-(\nu+1)}$ with $a(0) = 1$, respectively.

We now prove the validity of the connection process used in chapter III. We define $u_1(x,\varepsilon)$, $u_2(x,\varepsilon)$ in the neighborhood of $x = -1$, as the two functions obtained by the iteration procedures, multiplied by $\phi(x)$. These functions are continued analytically through the upper half plane above the singularities of equation (C) (see fig. A3). We use these solutions to solve the initial value problems at $x = \pm 1$ (in the outer regions), and obtain solutions that can be expressed as $a(u_2+\ell u_1)$, $A(u_2+ru_1)$ on the left and the right, with a, A, r, ℓ

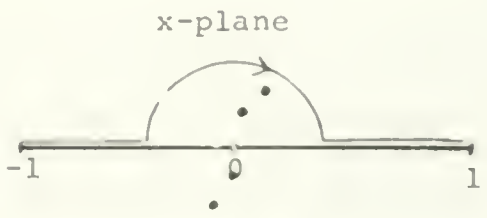

FIG. A3

depending on ε only. It is noted that $u_1(x,\varepsilon)$ is analytic with respect to ε, but that $u_2(x,\varepsilon)$ is only continuous in ε (for ν real) because, by its definition, v_o in (A10) is not analytic in ε. However, continuity when Re $\nu \neq -\frac{1}{2}$ suffices for our purposes. When u_1, u_2 are continued along the real axis from $x < 0$ to $x > 0$, we express the resulting functions in terms of the values of u_1, u_2 previously obtained, i.e. by connection coefficients. These coefficients have simple dependence on the connection coefficients $\alpha,\beta,\gamma,\delta$ of the functions $\varepsilon^{-\nu}u_1$, $\varepsilon^{\nu+1}u_2$. The last functions converge as $\varepsilon \to 0$ and z is held fixed, and therefore the connection coefficients associated with them, converge as $\varepsilon \to 0$ to the values that were used in the asymptotic treatment of chapter III. Finding complex eigenvalues requires that we solve the equation

$$\beta\tau^2 + (\ell\delta - r\alpha)\tau - r\ell\gamma = 0, \tag{A12}$$

where $\tau = \varepsilon^{i\mu}$ and all the coefficients are analytic functions of ε. (It is clearly necessary that ν be complex.) If instead of the coefficients in (A12) we take their value at $\varepsilon = 0$ (and keep the same notation), equation (A12) can be written as

$$\beta\tau^2 + (\ell\delta - r\alpha)\tau - r\ell\gamma = O(\varepsilon) \quad .$$

By a Rouché theorem argument, it is easy to show that the homogeneous and nonhomogeneous equations have the same number of zeroes.

CHAPTER IV: The Generalized Suydam Condition

IV.1 The Point of Overlap

We now use the results obtained in chapter III to decide the question of local stability near the point of overlap of the Alfvén and cusp continua. The inner region equation associated with this point, has four finite regular singularities, given by

$$(11) \qquad z_{1,2} = \frac{1}{-\alpha \pm \frac{1}{\sqrt{\rho}} h} \quad , \quad z_{3,4} = \frac{1}{-\alpha \pm \sqrt{\beta}\, h} \quad .$$

Assuming for simplicity $\alpha > 0$, three situations are possible:

i) $h^2 - \rho\alpha^2 < 0$; All four singularities are in the left half plane.

(34)

ii) $\beta h^2 < \rho\alpha^2 < h^2$; Only z_1 is in the right half plane.

iii) $\beta h^2 - \rho\alpha^2 > 0$; z_1, z_3 are in the right half plane, the other two points are in the left plane.

(α, β here are defined in (9) and are not connection coefficients).

If relation (34.i) holds, it was indicated before that the connection matrix is the identity matrix, and a sequence of complex eigenvalues cannot exist near $\varepsilon = 0$.

Next we deal with case (34.iii). From expression (10) we find the behavior of solutions of (D) near the singularities. Near the Alfvén singularity z_1 (and likewise z_2) two independent solutions can be chosen to be

$$u_1(z) \quad , \quad u_1(z_1) = 1$$

(35)

$$u_2(z) = u_1(z) \log (z-z_1) + f(z)$$

with u_1,f analytic functions in a neighborhood of z_1. Near the cusp singularity z_3 (and likewise z_4), there exists two solutions of the form

(36) $$v_1(z) \quad , \quad v_1(z_3) = 0 \quad , \quad v_1'(z_3) = 1$$

$$v_2(z) = v_1(z) \log (z-z_3) + g(z)$$

with v_1,g analytic near z_3.

Let $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = S \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a constant matrix. We now continue the solutions (35) along a path which winds around the two singularities z_1, z_3 (see fig. 7). Clearly $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix} S^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.
A simple calculation shows that the trace of this connection matrix is

(37) $\quad tr_1 = 2 + \dfrac{4\pi^2 b^2}{\Delta}$

where Δ is the determinant of S and b is its upper off diagonal element. By the definition, $u_1 = av_1 + bv_2$. Hence $b = \dfrac{[u_1,v_1]}{[v_2,v_1]}$ with the brackets again standing for the Wronskian. By the choice of the solutions, b is real and therefore b^2 is non negative. As to Δ, $[u_1,u_2] = \Delta[v_1,v_2]$. Since equation (D) is of a self adjoint form, $\tilde{p}[u_1u_2]$ = constant for any two solutions u_1, u_2 . We write

z-plane

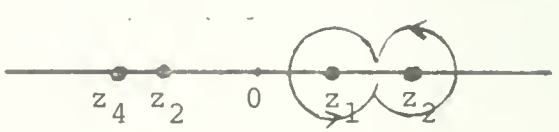

FIG. 7

(38) $$\Delta = \frac{\tilde{p}[u_1,u_2]}{\tilde{p}[v_1,v_2]}$$

and evaluate the numerator for $z \to z_1$, while the denominator is evaluated at $z \to z_3$.

$$[u_1,u_2] = [u_1,f] + \frac{u_1^2}{z-z_1} \quad .$$

By (10) $\tilde{p} = \frac{r}{K^2}A(z)$. As $z \to z_1$, $\tilde{p} \sim \frac{r}{K^2}A'(z_1)(z-z_1)$

and $\tilde{p}[u_1,u_2] = \frac{r}{K^2} A'(z_1)$. A is quadratic in z with a leading term $(h^2-\rho\alpha^2)z^2$, and z_1 is its rightmost zero. Hence $A'(z_1) > 0$ and the numerator of (38) is positive.

$$[v_1,v_2] = [v_1,g] + \frac{v_1^2}{z-z_3} \quad .$$

By (36) $\tilde{p}[v_1,v_2] = -\tilde{p}v_1'g(z_3)$. g satisfies the equation

$$(\tilde{p}g')' + \tilde{q}g = -2\frac{\tilde{p}v_1'}{z-z_3} - (\frac{\tilde{p}}{z-z_3})'v_1$$

and is a regular function near z_3. If q_o is the coefficient of $\frac{1}{z-z_3}$ in the expansion of $\tilde{q}$, we get $q_o g(z_3) = -\tilde{p}v_1'(z_3)$ or $-\tilde{p}v_1'g(z_3) = q_o g^2(z_3)$. As seen from (10), the sign of q_o = sign $AC'(z_3)$ which is clearly positive. Therefore in (38) $\Delta > 0$ and we have $tr_A > 0$. We now repeat a similar process for the evaluation of tr_2. The same argument follows through, where we have to decide the sign of $A'(z_2)A(z_4)C'(z_4)$. Clearly $A'(z_2) < 0$, $A(z_4) > 0$, $C'(z_4) < 0$, and again we get $tr_2 \geq 0$. Case (34.iii) is therefore in the category of (31.a), and accumulation of complex eigenvalues is possible regardless of the position of the boundary or the global properties of the plasma. Case (34.ii) is not as simple as the others and

its full details are yet to be worked out. Although in the right half plane we can use the two solutions (35) to determine easily that $tr_1 = 2$, in the left half plane we have three singularities and the previous methods fail to yield the sign of tr_2. It is instructive to mention that at least in some special cases, (34.ii) does yield results similar to (34.iii). In the limit $\beta \to 0$ (zero speed of sound) the two cusp singularities coalesce into a regular singularity located at $z = \frac{-1}{\alpha}$ in the left hand plane. If furthermore the equilibrium flow has no azimuthal component ($v = 0$), this point is a regular point of the equation. Accordingly, the only singularity in the left half plane is z_2 and $tr_2 = 2$. Because of the continuous dependence of tr_2 on β and v, tr_2 will remain positive for β and v small enough, so that (31.a) holds.

It would be useful to work out a special case in detail, and to convince ourselves of the added feature of non self adjoint systems, namely the global dependence of the angle of accumulation. The case we describe is incompressible flow, that is, a divergence free perturbed velocity field. This case can be reduced formally from the Lundquist equations (L) by letting the speed of sound approach infinity, which amounts to letting $B_*^2 \to \infty$, $\beta \to 1$. Proceeding with the asymptotic

expansion described in chapter III, we find that the inner region equation is:

$$(39)\quad (Au')' + [\frac{1}{A}\frac{4k^2}{r^2}(B_\theta H-\rho\sigma v)^2 - rK^2(\frac{1}{r}p_*^{o\prime})' - \frac{2m}{r}(B_\theta h-\rho v\alpha)]u = 0$$

The term containing the two cusp singularities disappeared, and we have only three regular singularities in the plane: the point at infinity and the two Alfvén singularities $z_{1,2} = \frac{1}{-\alpha \pm h/\sqrt{\rho}}$ which are in different sides of the imaginary axis if $h^2-\rho\alpha^2 > 0$. Assuming this inequality to hold, the indicial equation that determines the behavior (z^c, say) of solutions near $z_{1,2}$ is

$$c^2 + \frac{1}{A'^2}\frac{4k^2}{r^2}(B_\theta H - \rho\sigma v)^2 \Big|_{z_{1,2}} = 0 \quad .$$

If we let $\lambda_{1,2} = \frac{2k}{rA'}(B_\theta H-\rho\sigma v)\Big|_{z_{1,2}} = \frac{k(B_\theta \mp\sqrt{\rho}\, v)}{r(h\mp\sqrt{\rho}\,\alpha)}\Big|_{x=0}$

then $tr_{1,2} = 2\cosh(2\pi\lambda_{1,2})$.

By (32) we find that $\alpha + \delta e^{-\mu\pi} = \frac{4}{1+e^{\mu\pi}}\sinh\pi(\lambda_1+\lambda_2)\sinh\pi(\lambda_1-\lambda_2)$. This expression vanishes (corresponding to a 90° angle of accumulation) if and only if $|\lambda_1| = |\lambda_2|$, which clearly is not the general case. It should be noted that the incompressible limit is a regular limit for the connection coefficients, and that as a result we can reach a similar conclusion for a^2 large enough. The reason is that the

connection matrix is derived from solutions of the equation along a finite path which is bounded away from any singularity as $a^2 \to \infty$.

IV.2 The Tip of the Continuum

It was shown at the end of chapter II that beside the point of overlap, a tip of the continuum is the only possible accumulation point of unstable modes. We first consider a tip of the Alfvén continuum, given by

$$A(r_o,\omega_o) = A'(r_o,\omega_o) = 0 \quad .$$

Let $x = r-r_o$, then near $x = 0$ and for ω close to ω_o

$$A = H^2-\rho\sigma^2 = H^2-\rho\sigma_o^2-2\rho\sigma_o(\omega-\omega_o)+O(\omega-\omega_o)^2 \sim ax^2-2\rho\sigma_o(\omega-\omega_o)$$

where $$\sigma_o = \sigma(x,\ \omega=\omega_o)$$

$$a = \frac{1}{2}\,A''(x=0,\ \omega=\omega_o) \quad .$$

If $\left.\frac{2\rho\sigma}{a}\right|_{x=0} > 0$, the part of the Alfvén continuum with the tip ω_o is in the half line $\omega \geq \omega_o$. For simplicity we assume this to be the case and we define ε such that $\varepsilon^2 = \omega-\omega_o$ and $\mathrm{Im}\ \varepsilon \geq 0$. If $\frac{2\rho\sigma}{a} < 0$ we set $\varepsilon^2 = \omega_o-\omega$. Proceeding as in chapter III, we divide the interval of x into outer and inner regions corresponding to $|x| \gtrsim |\varepsilon|$ and $|x| \lesssim |\varepsilon|$. Using equation (B) to describe the plasma, we find that $\det A$ vanishes when A vanishes, and therefore when x and ε are small enough, the outer region equation

takes the form

$$[(x^2 - \frac{2\rho\sigma}{a}\varepsilon^2)u']' + qu = 0 \tag{40}$$

with the coefficients evaluated at $x = 0$, $\varepsilon = 0$. In the inner region we define $z = \frac{x}{\varepsilon}$, and for z fixed and $\varepsilon \to 0$, we get

$$[(z^2 - \frac{2\rho\sigma}{a}) \; u']' + qu = 0 \tag{41}$$

which is Legendre's equation. In contrast with the overlapping case, this equation has only 3 singularities in the complex z plane and not five of them. As often happens with special functions, we do not need the available explicit solution of equation (41), but derive the required results from the equation itself. As in (18) , a necessary condition to have an accumulating sequence of complex eigenvalues ε near $\varepsilon = 0$, is that $q > \frac{1}{4}$. Introducing the connection coefficients as before, we find that $\alpha+\delta e^{-\mu\pi} = 0$, since the coefficients of (41) depend on z^2 alone. We thus find that asymptotically, the complex ε eigenvalues are purely imaginary, corresponding to modes $\omega < \omega_o$ on the real axis. As assumed before, this ray is not part of the continuum and we have established that eigenvalues converging to ω_o must be real. This result

is not only asymptotic but exact, as can be derived from equation (B) by an oscillation argument as follows. For $\omega = \omega_o$ the equation is singular at $x = 0$. Also, if $q > \frac{1}{4}$ the solutions oscillate rapidly near the origin. Thus, for ω close to ω_o and real, but not in the spectrum ($\omega<\omega_o$), every solution of (B) has many zeroes(although not an infinite number) near $x = 0$, which is a consequence of the analytic dependence of the equation and its solutions on ω, when we consider $|x|>\delta>0$ with δ small as we wish. More precisely, we consider the solution that satisfies $u(-1) = 0$, $u'(-1) = 1$. As we let $\omega \to \omega_o$, zeroes of u appear at $x = 1$ and move into the interval $(-1,1)$. Every such zero yields an eigenvalue and the existence of many zeroes of the solution in the range of the plasma for ω close to ω_o, indicates the existence of a sequence of eigenmodes $\omega_n \to \omega_o$. The existence of a sequence of real eigenvalues converging at a tip of the continuum was discovered by Tataronis in the static equilibrium case. It was then shown [12] to be either Sturmian or anti Stumian, i.e. the number of nodal points increases or decreases monotonically as $\omega \to \omega_o$. It will be shown in chapter V that this property is lost in the steady flow equilibrium.

The tip of the cusp continuum can be treated in an identical way and yields similar results. We only have to

observe that det Λ vanishes when C vanishes, and that although the inner region equation is not the Legendre equation, it still depends on z^2 alone.

IV.3 A Necessary and Sufficient Criterion for Local Stability

We have established that only one point of the continuous spectrum can serve as an accumulation point of complex eigenvalues, and this is the overlapping point of the Alfvén and cusp continua. To achieve local stability, i.e. non existence of such a sequence, one of the following three conditions must be satisfied:

(i) The (finite) singularities of equation (C), for complex ω near ω_o (the point of overlap) are in one side of the real axis in the x plane. i.e.

$$h^2 - \rho\alpha^2 < 0 \tag{42}$$

(ii) A non-oscillation condition, requiring that when we set $\omega = \omega_o$, the real solutions of (C) do not oscillate near $x = 0$, i.e.

$$a < \frac{1}{4} \tag{43}$$

with a defined in (8).

(iii) If the two cusp singularities are in the same half plane, upper or lower, i.e. $\beta h^2 < \rho\alpha^2$, then either (31.b) or (31.c) are satisfied. If (31.b) holds, there will exist an accumulating sequence for some location of the boundary and not for others.

We now write these conditions in terms of the plasma profiles. We recall that the coefficients (10) of the equation were evaluated at the point were $\frac{m}{r} B_\theta + kB_z = 0$, which determines the ratio $\frac{m}{k}$ in terms of the magnetic field. We further notice that the coefficients (10) are homogeneous of degree zero in m and k, and thus are functions of $\frac{m}{k}$ only. It is therefore possible to eliminate the wave numbers from the equation, so that all the stability conditions can be expressed in terms of the plasma profiles only.

We define $\mu = \frac{B_\theta}{rB_z}$ which is essentially the pitch of the magnetic field. μ' is then the shear of the magnetic field, and after expressing

$$h = -kB_z \frac{\mu'}{\mu}$$

$$\alpha = -k\left(\frac{1}{\mu} \Omega' - w'\right)$$

condition (42) takes the form

$$B_z^2 \mu'^2 < \rho(\Omega' - \mu w')^2 \tag{44}$$

with $\Omega = \frac{v}{r}$ and $w = u_z$. The right hand side of (44) is essentially determined by the shear of the equilibrium flow, and we see that with large sheared flow, local

modes are impossible. In particular, every local instability can be stabilized by a sheared axial flow which, as we recall, can be introduced into every equilibrium state without changing the other quantities. The stabilizing effect of sheared flow can be explained by the plasma particles being carried away from each other, thus preventing localized modes from occurring.

We define the non dimensional quantities

$$\phi = \frac{\sqrt{\rho}\alpha}{h} = \frac{\sqrt{\rho}}{B_z \mu'} (\Omega' - \mu w')$$

$$\psi = \frac{\sqrt{\rho} v}{B_\theta}$$

Inequality (43) takes the form

$$\frac{1}{8} r B_z^2 (1-\phi^2)\left(\frac{\mu'}{\mu}\right)^2 - \phi\psi B_z^2 \frac{\mu'}{\mu} + \frac{1}{2} r^2 \frac{B^2}{B_\theta^2} (\rho\Omega^2)' + p'$$

$$(45) \qquad > \frac{2}{\beta-\phi^2} \frac{B_\theta^2}{r} \left[(1-\beta)\left[\frac{1}{4}\psi^4(1-\phi^2) - \psi^3\phi + \frac{3}{2}\phi^2\psi^2\right] + \beta\left[\frac{1}{2}\psi^2(1+\phi^2) - 2\phi\psi + \phi^2\right]\right]$$

When we let the flow vanish, (45) reduces to

$$(46) \qquad \frac{1}{8} r B_z^2 \left(\frac{\mu'}{\mu}\right)^2 + p' > 0$$

which is Suydam's condition [10]. Condition [44] can be written as $\phi^2 > 1$ and therefore (45) may be considered only for $\phi^2 < 1$. If $\phi^2 = 1$ or $\phi^2 = \beta$, our analysis fails

and the singularity at infinity of the inner region equation, becomes an irregular singularity. The analysis also fails in the case of a zero shear magnetic field ($\mu' = 0$), which requires a separate treatment.

An interesting case is when the flow is purely axial. i.e., $\psi = \Omega = 0$. In this case (45) reduces to

$$(47) \qquad \frac{1}{8} r B_z^2 \left(\frac{\mu'}{\mu} \right)^2 (1-\phi^2) + p' > \frac{2\beta\phi^2}{\beta-\phi^2} \frac{B_\theta^2}{r}$$

When the shear of the flow is small enough so that $\beta - \psi^2 > 0$, (47) is more difficult to satisfy, compared with (46). Thus the introduction of flow into the equilibrium can cause local instability, although large flow shear makes the plasma locally stable.

CHAPTER V: The Lagrangian Description

V.1 The Equations of Motion

The Lagrangian formulation of the ideal MHD equations "near" a steady state equilibrium, was carried out by Frieman and Rotenberg [13] and, as will become apparent shortly, turns out to be superior to the Eulerian description wherever stability analysis is involved.

The position vector $\underline{r}$ of a plasma particle which was at $\underline{r}_o$ at $t = 0$, is written as

$$\underline{r} = \underline{r}^o + \underline{\xi}(\underline{r}^o, t) \tag{49}$$

where $\underline{r}^o$ describes the equilibrium trajectory of the same particle (located at $\underline{r}_o$ initially), and $\underline{\xi}(\underline{r}^o, t)$ is the displacement from equilibrium - assumed to be small. The independent variables are taken to be $\underline{r}^o$, t rather than $\underline{r}_o$,t so that the equilibrium quantities are time independent. After linearization, the conservation of mass, entropy and magnetic flux, yield to first order in $\underline{\xi}$:

$$\rho(\underline{r}^o + \underline{\xi}) = \rho(\underline{r}^o)(1 - \underline{\nabla}^o \cdot \underline{\xi})$$

$$p(\underline{r}^o + \underline{\xi}) = p(\underline{r}^o) - \rho a^2 \, \underline{\nabla}^o \cdot \underline{\xi}$$

$$\underline{B}(\underline{r}^o + \underline{\xi}) = \underline{B}(\underline{r}^o) - \underline{B} \, \nabla^o \cdot \underline{\xi} + \underline{B} \cdot \underline{\nabla}^o \, \underline{\xi}$$

where $\underline{\nabla}^{o}$ is the gradient operator in the $\underline{r}^{o}$ coordinates. The displacement vector $\underline{\xi}$ satisfies the equation

$$\rho\underline{\xi}_{tt} + 2A\underline{\xi}_{t} + H\underline{\xi} = 0 \tag{50}$$

where

$$A\underline{\xi} = \rho\underline{u}\cdot\nabla\underline{\xi}$$

$$-H\underline{\xi} = \nabla(\rho a^{2}\nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla p - \underline{B}\cdot\underline{Q}) + \underline{B}\cdot\underline{\nabla}\,\underline{Q}$$

$$+\underline{Q}\cdot\underline{\nabla}\,\underline{B} + \underline{\nabla}\cdot(\rho\underline{\xi}\,\underline{u}\cdot\underline{\nabla}\,\underline{u} - \rho\underline{u}\,\underline{u}\cdot\underline{\nabla}\,\underline{\xi})$$

$$\underline{Q} = \operatorname{curl}(\underline{\xi}\times\underline{B})$$

All coefficients are the unperturbed quantities and are functions of $\underline{r}^{o}$ only. The superscript was dropped everywhere for simplicity.

Equation (50) is an initial value problem and the proper values to prescribe are $\underline{\xi}, \underline{\xi}_{t}$ at $t = 0$. A well posed initial value problem for the Ludquist equations (L) requires, that we give the values of $\rho, p, \underline{u}, \underline{B}$ initially. It is clear even from counting the number of data, that the Eulerian and Lagrangian descriptions are not entirely equivalent. It was pointed out to the author by H. Grad, that the class of flows resulting from

prescribing the Lagrangian data, is contained (in the strong sense) in the Eulerian flows. An illustrative example is the "frozen in" field equation $\underline{B}_t + \text{curl}(\underline{B}_o \times \underline{u}) = 0$, where $\underline{B}_o$ is a static equilibrium field, and $\underline{B}$, $\underline{u}$ are the perturbations. A change to Lagrangian variables requires that we write $\underline{u} = \underline{\xi}_t$ and integrate the equation with respect to time. The result: $\underline{B} + \text{curl}(\underline{B}_o \times \underline{\xi}) = [\underline{B} + \text{curl}(\underline{B}_o \times \underline{\xi})]_{t=0}$. The right hand side is always dropped from the equation, but this imposes a (non-holonomic) constraint on the flow, namely $\underline{B}(t=0) = \text{curl}(\underline{\xi}(t=0) \times \underline{B}_o)$. An important outcome of this constraint is that an unperturbed system with closed line magnetic field, will remain so under the Lagrangian perturbation, while the Eulerian data allow the freedom of changing the topological character of the field lines.

We return to the question of stability of equation (50). By choice of the independent variables, time does not appear explicitly in (50) and the equation can be Laplace transformed in time, or equivalently we can assume an $e^{i\omega t}$ dependence. Thus, (50) takes the form

$$-\omega^2 \rho \underline{\xi} + 2\omega i A \underline{\xi} + H\underline{\xi} = 0 \tag{51}$$

A natural Hilbert space setting for this problem can be obtained by introducing the scalar product

$$(\xi,\eta) = \int \underline{\eta}^* \cdot \underline{\xi}\, dV$$

with the integration carried over the volume of the plasma, and $\underline{\eta}^*$is the transposed and complex conjugate vector of $\underline{\eta}$. It can be easily shown that for every $\underline{\xi}$ whose normal component vanishes at the boundary of the plasma, the quadratic forms $(H\xi,\xi)$ and $(iA\xi,\xi)$ are real. This proves that the operators H and iA are symmetric (formally self adjoint). We can see now how non self adjointness sets in when flow is introduced into the equilibrium state. Without flow, $A = 0$ and (51) takes the form of a classical eigenvalue problem with the result that ω^2 must be real. In the presence of flow, no such simple result is available, although regions in the complex plane free of "eigenvalues" ω, can be found without much labor. There is a considerable amount of literature dealing with the non-linear eigenvalue problem (51), and an extensive reference list can be found in reference 14. Most of the results deal with variational properties of the eigenvalues, completeness of the eigenvectors, etc., but always under restrictive assumptions which assure the reality of ω(e.g. H is positive), or deal with finite dimensional spaces.

The stability question received less attention, but from what is known it seems highly unlikely that a simple criterion for stability can be found for the general case. As an illustration to the difficulty encountered, we state the condition derived by Barston [15] for the stability (boundedness in time) of the system $u_{tt} + Au_t + Hu = 0$ in an n-dimensional Eucledian space, with H and iA Hermitian, constant matrices. The system is stable if and only if there exists a positive definite Hermitian matrix p and an anti-Hermitian matrix α, such that $A = p\alpha + \alpha p$, $H = p(\alpha^2 + \frac{1}{4})p$. Although such a condition is useful for constructing stable systems, it is practically useless as a stability test.

We continue now with the special case of the screw-pinch configuration. Initially, the equilibrium profiles are assumed to depend on r_o alone. Since the radial velocity vanishes, we have $r^o = r_o$ and the coefficients of (51) do not depend on θ^o, z^o. We Fourier transform the equations with respect to these variables, and get

$$\underline{\xi}(\underline{r}^o, t) = \underline{\xi}(r_o) \exp i(m\theta^o + kz^o + \omega t)$$

We can express $\theta^o = \theta_o + \Omega t$, $z^o = z_o + wt$, and in

terms of the initial location of the fluid particle, we have

$$\underline{\xi}(\underline{r}^o,t) = \underline{\xi}(r_o) \exp i(m\theta_o + kz_o)e^{i\sigma t}$$

where as before $\sigma = \omega+\underline{k}\cdot\underline{u}_o$. According to the usual Lagrangian procedure, the velocity field is obtained by differentiating $\underline{r}$ with respect to time, keeping $\underline{r}_o$ fixed. By differentiating (49) we get

$$\underline{u}(\underline{r},t) = \underline{u}_o(\underline{r}^o)+(\underline{u}_o\cdot\underline{\nabla}^o)\underline{\xi} + \underline{\xi}_t\Big|_{\underline{r}_o \text{ fixed}}$$

here $\underline{u}_o$ is the equilibrium flow and $\underline{u}$ is the total flow at $\underline{r}$ (not $\underline{r}^o$). By a Taylor expansion we have $\underline{u}(\underline{r}) = u(\underline{r}^o)+(\underline{\xi}\cdot\nabla^o)\underline{u}_o$ to first order, and therefore

$$\delta\underline{u} = \underline{u}(\underline{r}^o)-\underline{u}_o(\underline{r}^o) = i\sigma\underline{\xi}-\xi_r(r\overset{o}{\Omega}\overset{o}{w}) \tag{52}$$

where Ωr , w are the angular and axial equilibrium flow. $\delta\underline{u}$ is the velocity perturbation, previously called $\underline{u}$, and (52) provides the transition from Lagrangian to Eulerian analysis. In particular, the radial components are related simply by $\delta u_r = i\sigma\xi_r$, which was used in the derivation of equation (B). The conclusion is that (51) can be reduced to a single second order differential equation in ξ_r , which is exactly equation (B).

In view of future benefits we will nevertheless re-derive this equation, using the Lagrangian formulation only.

We rewrite equation (51) in the form

$$H_\omega \underline{\xi} = 0 \tag{53}$$

where $$H_\omega = -\rho\omega^2 + 2\omega i A + H$$

claim: For ω real, equation (53) has the equivalent variational formulation

$$\delta(H_\omega \xi, \xi) = 0$$

where the variation is with respect to all twice differentiable complex vectors $\underline{\xi}$, with a vanishing normal component on the boundary of the plasma.

proof: For ω real, H_ω is a symmetric operator.

$$\delta(H_\omega \xi, \xi) = (H_\omega \delta\xi, \xi) + (H_\omega \xi, \delta\xi) = 2\,\mathrm{Re}(H_\omega \xi, \delta\xi) = 0.$$

By taking $i\delta\xi$ instead of $\delta\xi$ we find $\mathrm{Im}(H_\omega \xi, \delta\xi) = 0$. Thus $H_\omega \underline{\xi} = 0$.

In the screw pinch case $(H_\omega \xi, \xi)$ contains r-derivatives of ξ_r only, and is algebraic in $\xi_\theta, \xi z$. Moreover,

if we let

$$\xi = \xi_r \ , \ \eta = i\xi_\theta \ , \ \zeta = i\xi_z$$

then the coefficients of the quadratic form are real, and (53) has a real solution. We thus consider ξ,η,ζ to be real and view $(H_\omega\xi,\xi)$ as an algebraic quadratic in η,ζ . This quadratic has the following terms: Second degree terms in η, ζ:

$$(54) \quad S(\eta,\zeta) = \rho a^2(\tfrac{m}{r}\eta + k\zeta)^2 + K^2(\eta B_z - \zeta B_\theta)^2 - \rho\sigma^2(\eta^2+\zeta^2)$$

First degree terms:

$$\tfrac{2}{r}(\tfrac{m}{r}\eta + k\zeta)[\rho a^2(r\xi)' + \rho v^2\xi] + 2(\eta B_z - \zeta B_\theta)[\tfrac{1}{r}(kB_\theta + \tfrac{m}{r}B_z) + (\tfrac{m}{r}B_z - kB_\theta)\xi']$$

$$-4\rho\Omega\sigma\xi\eta$$

Zeroeth degree terms:

$$B_*^2\,\xi'^2 + \xi^2[H^2 - \rho\sigma^2 - (\tfrac{\rho a^2}{r})' + \frac{B_z^2}{r^2} - \frac{B_\theta^2}{r^2}]$$

This quadratic is integrated along the range of the plasma with the measure rdr. The terms containing $\xi\xi'$ were eliminated by integration by parts. The terms containing η,ζ can be written as $I(y) = (Sy,y) + 2(b,y)$

where $y = \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$, S is the matrix associated with (54), and b is a vector linear in ξ, ξ'. I(y) is an algebraic quadratic and it is stationary for y_o, where

$$y_o = -S^{-1}b \quad . \tag{55}$$

Substituting these values of η, ζ we get

$$(H_\omega \xi, \xi) = \int (p\xi'^2 - q\xi^2)\, dr$$

and the corresponding Euler-Lagrange equation is

$$(p\xi')' + q\xi = 0 \quad .$$

The coefficients p,q are exactly those defined in (B). The substitution (55) was instrumental in reducing the original equation (53) to the scalar equation (B) in ξ alone. Although the reduction was based on the assumption that ω was real, we know that the coefficients of (53) are analytic in ω, and thus we can conclude that equation (B) applies to ω complex too, as is known from the Eulerian description.

The matrix S has the determinant

$$\det S = \rho^2\sigma^4+CK^2 = \lambda_{12}$$

as defined in (B), and it is now clear how this apparent singularity enters into (B). From the discussion in section II.2 we know already that ξ has no singularity at the point where $\lambda_{12} = 0$. As to η,ζ, we use (52) to express them in terms of the Eulerian variables which are not singular at this point, and we conclude that λ_{12} is not part of the spectrum of the Lagrangian equation, despite (55). We also find that $\sigma = 0$ is not part of the spectrum in this formulation. This difference between the Eulerian and Lagrangian spectra is not surprising in view of the discussion at the beginning of the chapter. The continuous spectrum is related to solving an initial value problem, and this problem is not equivalent in the two descriptions.

V.2 Properties of the Point Spectrum

We take the inner product of (51) with $\underline{\xi}$ and solve for ω.

$$(56)\quad \omega = \frac{1}{(\rho\xi,\xi)}\left[(iA\xi,\xi) \pm \sqrt{(iA\xi,\xi)^2+(H\xi,\xi)(\rho\xi,\xi)}\right]$$

One observes [13] that a sufficient condition for stability (ω real), is

$$(57)\qquad (iA\xi,\xi)^2 + (H\xi,\xi)\ (\rho\xi,\xi) \geq 0$$

for all ξ. This is known as the overdamping condition. If ω is a complex eigenvalue, we have

$$(58)\qquad \mathrm{Re}\ \omega = \frac{(iA\xi,\xi)}{(\rho\xi,\xi)}$$

$$(59)\qquad |\omega|^2 = \frac{-(H\xi,\xi)}{(\rho\xi,\xi)}$$

where $\underline{\xi}$ is the corresponding eigenfunction. These results were derived by Barston [16]. It was also observed by the same author that if H is bounded below, say $(H\xi,\xi) \geq \lambda(\rho\xi,\xi)$ for all $\underline{\xi}$ (understood to have a vanishing normal component on the boundary) and with $\lambda > -\infty$, we have

$$(60)\qquad |\omega|^2 \leq -\lambda$$

This condition restricts the complex eigenvalue to a circle in the complex plane, and what is more important from the point of view of stability, it provides an upper bound for the growth rate of instabilities. A better bound for the growth rates can be derived when A is a bounded operator (as will be shown to be the case in the screw pinch equilibrium). We make a real translation of the complex ω plane

$$\omega = \alpha + \omega'$$

with a real constant α. We get

$$H_\omega = -\rho\omega'^2 + 2\omega'(iA-\alpha\rho)+H_\alpha$$

where H_α is defined in (53). The last expression is similar in form to $H_{\omega'}$, and using an argument similar to (60) we can conclude

$$(61) \qquad |\omega'|^2 \leq -\lambda_\alpha$$

$$\lambda_\alpha = \operatorname*{Inf}_{\xi} \frac{(H_\alpha\xi,\xi)}{(\rho\xi,\xi)}$$

One can pick an optimal α by maximizing λ_α. We then get

$$(62) \qquad (\text{Im } \omega)^2 \leq -\operatorname*{Max}_{\alpha \text{ real}} \operatorname*{Inf}_{\xi} \frac{(H_\alpha\xi,\xi)}{(\rho\xi,\xi)}$$

for any complex eigenvalue of (53). Our new bound enables us to state the following

Corollary: A sufficient condition for stability is that the following holds:

$$(63) \qquad \underset{\alpha \text{ real}}{\text{Max}} \ \underset{\underline{\xi}}{\text{Inf}} \ \frac{(H_\alpha \zeta, \xi)}{(\rho\xi, \xi)} \geq 0 \quad .$$

The next section will be devoted to the interpretation and investigation of this criterion. We now show that in the screw pinch equilibrium, the last criterion is applicable(i.e. the previous assumptions are satisfied).

Claim: A is bounded, H is bounded below.

Proof: BY the definition of A

$$(64) \qquad A\underline{\xi} = \rho \underline{u}\cdot\nabla \ \underline{\xi} = i\rho(\underline{k}\cdot\underline{u})\underline{\xi} + \rho\Omega(-\xi_\theta \underset{0}{r})$$

A is not a differential operator, and $||A|| \leq \underset{r}{\text{Max}} \ \rho[|\underline{k}\cdot\underline{u}|+|\Omega|]$. Changing our notation from α to ω(real), we now want to show that H_ω is bounded below. The claim refers to the special case $\omega = 0$. From the explicit expression of $(H_\omega\xi,\xi)$ detailed in section V.1, we see that this quadratic can be viewed as an algebraic quadratic in ξ', having as first and second degree terms the expression

$$(65) \qquad J(\xi') = B_\star^2 \ \xi'^2 + 2\xi'[\rho a^2(\frac{m}{r}\eta + k\zeta) + (kB_\theta - \frac{m}{r}B_z)(B_\theta\zeta - B_z\eta)]$$

All other terms are of second degree in ξ,η,ζ. The coefficients of the quadratic are real and minimization can be carried out by considering only real vectors (ξ,η,ζ). The reason: if $(H_\omega\xi,\xi) \geq \lambda_\omega(\rho\xi,\xi)$ for all real ξ, then

$$(H_\omega(\xi+i\phi),\xi+i\phi) = (H_\omega\xi,\xi)+(H_\omega\phi,\phi)-2\ \mathrm{Im}(H_\omega\xi,\phi) = \lambda_\omega[(\rho\xi,\xi)+(\rho\phi,\phi)]$$

because $\mathrm{Im}(H_\omega\xi,\phi) = 0$ when ξ and ϕ are real. Similarly $(\rho(\xi+i\phi),\ \xi+i\phi) = (\rho\xi,\xi)+(\rho\phi,\phi)$, and we see that the same bound applies to real and complex vectors. We therefore assume ξ,η,ζ to be real, and minimize (65) with respect to ξ'. The minimum value of $J(\xi')$ is

$$(66) \qquad -\frac{1}{B_*^2}\left[\rho a^2\left(\frac{m}{r}\eta+k\zeta\right)+\left(kB_\theta-\frac{m}{r}B_z\right)(B_\theta\zeta-B_z\eta)\right]^2 .$$

(It is assumed here, as in section II.2, that B_*^2, ρa^2 do not vanish in the range of the plasma). To find a lower bound for $(H_\omega\xi,\xi)$ we replace $J(\xi')$ by (66), and get an algebraic quadratic in ξ,η,ζ. Such a quadratic is bounded below by the lowest eigenvalue of the associated matrix and the claim is proved.

We carry out this minimization in order to obtain a lower bound for λ_ω, and thus an upper bound for the growth rates. After simple algebraic manipulations we write the expression as a sum of two quadratic forms, the first of

which can be represented by the symmetric matrix Q with elements

$$Q_{11} = [r(\frac{B_\theta^2}{r^2})' - (\frac{B_*^2}{r})' + H^2]/[1 + \sqrt{1 - \frac{H^2}{K^2B^2}}]$$

$$Q_{12} = \frac{K}{r}(\rho a^2+\rho v^2)/[1 + \sqrt{1 - \frac{H^2}{K^2B^2}}]\frac{1}{2}$$

(67)
$$Q_{13} = \frac{B}{r}(kB_\theta + \frac{m}{r}B_z)/[1 + \sqrt{1 - \frac{H^2}{K^2B^2}}]\frac{1}{2}$$

$$Q_{22} = \beta B^2K^2$$

$$Q_{23} = \beta BK(kB_\theta - \frac{m}{r}B_z)$$

$$Q_{33} = (1-\beta)H^2+\beta B^2K^2$$

where Q is represented in the space of vectors $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with

(68) $$x = [1+\sqrt{1 - \frac{H^2}{K^2B^2}}]^{\frac{1}{2}}\xi \quad , \quad y = \frac{1}{K}(\frac{m}{r}\eta + k\zeta) \quad , \quad z = \frac{1}{B}(\eta B_z - \zeta B_\theta)$$

We have

$$(H_\omega \xi, \xi) \geq (Qx,x) - \rho[\sigma^2(\xi^2+\eta^2+\zeta^2)+4\Omega\sigma\xi\eta]$$

The minimum of the right hand side would not increase if we minimize the two terms separately. From (68) we find

$$(1-\sqrt{1 - \frac{H^2}{K^2B^2}}\ (\eta^2+\zeta^2) \leq y^2+z^2 \leq (1+\sqrt{1 - \frac{H^2}{K^2B^2}}\ (\eta^2+\zeta^2)$$

(which is responsible for the definition of x). Thus, if q is the lowest eigenvalue of Q (and we assume it to be negative at least for same r in the plasma), we find

$$(69) \qquad \inf_{\underline{\xi}} \frac{(H_\omega \xi, \xi)}{(\rho\xi, \xi)} \geq \min_r \left[\frac{q}{\rho}\left(1+\sqrt{1-\frac{H^2}{\kappa^2 B^2}}\right) - (\sigma^2+2|\sigma||\Omega|)\right]$$

If q is positive everywhere, we replace the radical term by its negative value. The minimum over r is taken, because a vector $\underline{\xi}$ can be concentrated around the point where the minimum is attained. Expression (69) has to be optimized with respect to all real ω. It is possible to save much effort by choosing $\omega = \omega_o$ such that $\max_r(\sigma^2+2|\sigma||\Omega|)$ be minimized by ω_o.

Relation (69) provides us with the following result. If K is large enough and $H \neq 0$ in the range of the plasma, then the matrix Q is positive definite and we get

$$(70) \qquad (\text{Im } \omega)^2 \leq \min_\omega \max_r (\sigma^2+2|\sigma||\Omega|)$$

for every complex eigenvalue ω.

Before continuing the discussion about the best bound λ_ω, we find the range of Re ω for complex eigenvalues. From (64) we have

$$(iA\xi, \xi) = -\int \rho[(\underline{k}\cdot\underline{u})(\xi^2+\eta^2+\zeta^2)+2\Omega\xi\eta]rdr.$$

The eigenvalues of the quadratic under the integral sign are $-\underline{k}\cdot\underline{u}$, $-\underline{k}\cdot\underline{u}\pm\Omega$, and by (58) we get

$$(71) \qquad \underset{r}{\text{Min}}[-\underline{k}\cdot\underline{u} - |\Omega|] \leq \text{Re } \omega \leq \underset{r}{\text{Max}}\ [-\underline{k}\cdot\underline{u} + |\Omega|]$$

As a demonstration we use the results just established, for the special case of solid rotation, i.e. Ω = constant, w = 0. In this case $\sigma = \omega+m\Omega$ and the optimal ω for inequality (69) is $\omega_o = -m\Omega$, so that $\sigma(\omega_o) = 0$. Relation (70) yields stability for large K (and $H\neq 0$). In the case of instability we find from (71)

$$-|\Omega| \leq \text{Re } \omega + m\Omega \leq |\Omega|$$

We return now to the optimization (with respect to ω) of λ_ω. If H_ω was a self adjoint operator (and not only formally self adjoint), we could have concluded that λ_ω was the lowest point in the (λ) spectrum of H_ω, characterized by the equation

$$(72) \qquad H_\omega \underline{\xi} = \lambda \rho \underline{\xi}$$

Nevertheless, our purpose is to suggest a candidate ω_o for optimization and not to actually find λ_{ω_o}, and to this end we can treat H_ω as a self adjoint operator.

The continuous spectrum of H_ω is easily detected. We repeat the procedure carried out at the beginning of the chapter in reducing (53) to (B). The only difference is that ω^2 is replaced by $\omega^2+\lambda$. We thus reduce (72) to a second order ODE similar to (B), and the continuous spectrum exposes itself through the vanishing of

$$\tilde{A} = H^2-\rho\sigma^2-\rho\lambda \ , \quad \tilde{C} = \beta H^2-\rho\sigma^2-\rho\lambda$$

We find that the continuous spectrum is composed of all points

$$\lambda = \frac{1}{\rho} H^2 - \sigma^2 \text{ or } \lambda = \frac{\beta}{\rho} H^2 - \sigma^2$$

for some r in the plasma . The lowest point of the continuous spectrum is clearly $\lambda_c(\omega) = \min_r(\frac{\beta}{\rho} H^2-\sigma^2)$. If there is not any eigenvalue lower than λ_c (for all ω), we can choose ω_o to be the ω maximizing λ_c. We now assume that there exists an eigenvalue $\lambda_o < \lambda_c$. From (B) we know that every eigenvalue λ is an analytic function of ω as long as $\lambda(\omega)$ does not cross the continuous λ-spectrum. Another observation we make is, that the eigenvalue curves $\lambda(\omega)$ do not cross each other. The reason: at a crossing point there will be two eigenfunctions for one

λ, which is impossible because λ is determined by a second order differential equation. As a result we can conclude that the lowest eigenvalue $\lambda_o(\omega)$ is an analytic function of ω. We can now differentiate (72) with respect to ω.

$$\dot{H}_\omega \underline{\xi} + H_\omega \dot{\underline{\xi}} = \dot{\lambda}_o \rho \underline{\xi} + \lambda_o \rho \dot{\underline{\xi}} \tag{73}$$

where $\dot{x} \equiv \frac{\partial x}{\partial \omega}$. Multiplying (73) by $\underline{\xi}$, and using the fact that $\dot{\underline{\xi}}$ satisfies the homogeneous boundary conditions (B1-B2), we get

$$(2\omega+\dot{\lambda}_o)(\rho\xi,\xi) = 2(iA\xi,\xi) \tag{74}$$

We are interested in the optimal $\lambda_o = \lambda_o(\omega_o)$, and we find from (74)

$$\omega_o = \frac{(iA\xi,\xi)}{(\rho\xi,\xi)} \quad .$$

Similarly to (71), we get

$$\min_r[-\underline{k}\cdot\underline{u} - |\Omega|] \leq \omega_o \leq \max_r [-\underline{k}\cdot\underline{u} + |\Omega|] \tag{75}$$

which is expected in view of (71). We want to show that if $\lambda_o = \lambda_\omega$, then there exists a unique optimal ω_o.

It would be sufficient to show that $\ddot{\lambda}_o < 0$. Differentiating (73) again, we get

$$\ddot{H}_\omega \underline{\xi} + 2\dot{H}_\omega \dot{\underline{\xi}} + \ddot{\underline{\xi}} = \ddot{\lambda}_o \rho \underline{\xi} + 2\rho\dot{\lambda}_o \dot{\underline{\xi}} + \lambda_o \rho \ddot{\underline{\xi}}$$

Multiplying by $\underline{\xi}$, and using the relation $H_\omega = -2\rho$, we find

$$4((iA-\omega\rho)\xi,\xi) - 2(\rho\xi,\xi) = \ddot{\lambda}_o(\rho\xi,\xi)+2\dot{\lambda}_o(\rho\dot{\xi},\xi)$$

We now multiply (73) by $2\dot{\underline{\xi}}$ and subtract it from the last equation

$$\ddot{\lambda}_o(\rho\xi,\xi) = -2(\rho\xi,\xi) + 2\lambda_o(\rho\dot{\xi},\dot{\xi})-2(H_\omega\dot{\xi},\dot{\xi})$$

If indeed $\lambda_o = \lambda_\omega$, then the last two terms give a negative contribution, and we conclude that $\ddot{\lambda}_o \leq -2$. The optimizing ω_o is thus unique.

The next section will offer a simple way to compute the lowest eigenvalue of H_ω numerically. The analytical results we derived here can thus have a practical use.

V.3 A Sufficient Condition for Stability

In the previous section we derived a sufficient condition for stability:

$$\text{(63)} \qquad \underset{\omega \text{ real}}{\text{Max}} \quad \underset{\xi}{\text{Inf}} \frac{(H_\omega \xi, \xi)}{(\rho\xi, \xi)} \geq 0 \ .$$

In the static equilibrium case $H_\omega = H - \rho\omega^2$, and the maximum with respect to ω is obtained for $\omega = 0$. The resulting sufficient condition for stability is $H \geq 0$, which is known to be a necessary condition as well. This is not the case when flow is present.

Condition (63) was originally derived (using a different method than ours) by Bussard [17], and was believed by him to be a necessary and sufficient condition for stability. The proof given by this author applied only to finite dimensional spaces, and of course the proof of the necessity part was erroneous. Bussard's idea however, offers a visual interpretation of condition (63) as well as a demonstration of its unnecessity. Let us then consider the case $\rho = 1$ and iA, H symmetric constant matrices of order n. The Eigenvalue equation $H_\omega \xi = 0$ has now exactly 2n solutions, given by $\det(H_\omega) = 0$. The system would be stable if we can demonstrate the existence of 2n real eigenvalues. To this end we consider the

λ-eigenvalue problem

$$H_\omega \xi = \lambda(\omega)\xi$$

By standard theorems [18] we know, that the n eigenvalues can be described by n functions $\lambda_k(\omega)$ $(k=1,\ldots n)$ which are analytic in a neighborhood of the real line. Starting at $\omega = 0$, for example, we label the eigenvalues $\lambda_k(0)$ according to the natural order, with λ_1 the smallest one. As we let ω vary along the real axis, a curve $\lambda_k(\omega)$ may cross the horizontal axis (in the (ω,λ) plane). The point of intersection is an ω-eigenvalue because the equation $H_\omega \xi = 0$ has a non-trivial solution. All real eigenvalues ω are given by such points of intersection. The curves $\lambda_k(\omega)$ behave like $-\omega^2$ for $|\omega|$ large and thus are negative. If for some ω λ_k becomes positive, there will be at least two points of intersection. If all of the curves become positive (fig. 8) we clearly have 2n real ω-eigenvalues and the system is stable. Condition (63) states that if the lowest λ-eigenvalue becomes positive for same ω, we have stability. Two counter examples would show why this

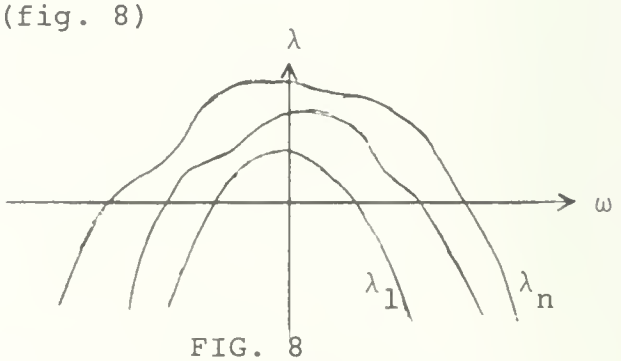

FIG. 8

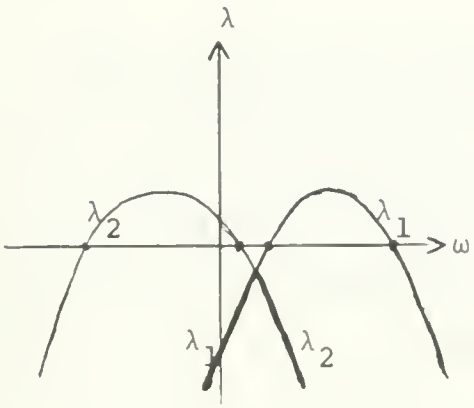

FIG. 9

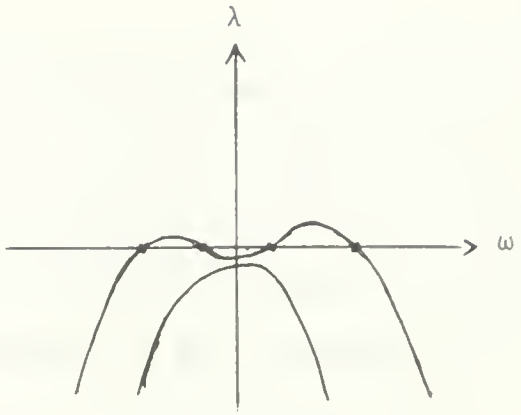

FIG. 10

condition is not necessary. In (fig. 9) the eigenvalue curves intersect. The lowest eigenvalue for any given ω is always negative, while the analytic curves $\lambda_k(\omega)$ intersect the ω axis. Parenthetically we add that in the screw pinch case this situation does not occur. The second counter example is when one of the higher curves crosses the axis more than twice (fig. 10), where we can still have stability although $\lambda_1(\omega)$ is always negative.

We return now to the minimization of $\frac{(H_\omega \xi, \xi)}{(\rho\xi, \xi)}$ for fixed ω. From the previous section we know that there is hope to find H_ω non-negative only if $\beta H^2 - \rho\sigma^2 \geq 0$ for all r in the plasma. We derive the same result in a more direct way by actually minimizing $\frac{(H_\omega \xi, \xi)}{(\rho\xi, \xi)}$. In fact, because we are only interested in the non-negativity of H_ω, we can take the infemum of $(H_\omega \xi, \xi)$ without normalizing ξ.

Like (54), $(H_\omega \xi, \xi)$ is an algebraic quadratic in η, ζ and we first minimize it with respect to these two

variables. The minimum is finite if and only if the second degree term $S(\eta,\zeta)$ is positive definite, or if it is semi definite, then the first degree term also vanishes when the second degree term does so. The positivity of S requires that

$$\text{trace } S = K^2B_*^2 - 2\rho\sigma^2 > 0 \tag{76}$$

$$\det S = \rho^2\sigma^4+K^2B_*^2(\beta H^2-\rho\sigma^2) > 0$$

be satisfied for all r. From the two inequalities it follows that $H^2-\rho\sigma^2 > 0$.

Assuming S to be positive definite, we minimize with respect to η,ζ and get the known form

$$F(\xi) = \int (p\xi'^2 - q\xi^2)\, dr \tag{77}$$

with p,q defined in (B). $F(\xi)$ is bounded below only if $p \geq 0$ everywhere in the plasma. Rewriting p as

$$p = \frac{rB_*^2(H^2-\rho\sigma^2)(\beta H^2-\rho\sigma^2)}{\det S}$$

it is clear that in addition to (76) we need the condition

$$\beta H^2 - \rho\sigma^2 \geq 0 \tag{78}$$

to hold everywhere. Inequality (78) is more restrictive than (76) (unless $H = \sigma = 0$) and it is sufficient to consider only it. ω can still be chosen arbitrarily as long as (78) holds, but if $H = 0$ at some point r_o , we must choose $\omega = \omega_o$, so that $\sigma(r_o,\omega_o) = 0$. The point (r_o,ω_o) is recognized as the point of overlap of the Alfvén and cusp continua. If there are two points in the plasma where $H = 0$, we cannot use the sufficient condition unless σ vanishes at both points for the same ω. This will be the case for instance, if the equilibrium velocity field is parallel to the magnetic field, up to a solid rotation.

Before we continue, a certain difficulty has to be removed. If the point of overlap of the continua has to be chosen in order to implement the sufficient condition, the strict condition $\det S > 0$ is violated at that point. Luckily however, the first degree term in (54) also vanishes at this point and minimization with respect to η, ζ does not yield $-\infty$.

The form $F(\xi)(p\geq 0)$ in (77) is non-negative for all ξ satisfying the homogeneous boundary conditions (B1-B2), if and only if the following condition holds:

i) If p is strictly positive everywhere, then no solution of the Euler-Lagrange equation (B) associated with $F(\xi)$, vanishes twice in the domain of the plasma [19].

ii) If p vanishes at r_o corresponding to the overlap of the continua, but otherwise is positive (except at r = 0), then no solution of (B) vanishes twice in any of the subintervals $(0,r_o),(r_o,R)$ [11].

It is a simple task to find numerically whether or not this condition is satisfied. We can use this criterion to compute λ_ω of (61) numerically. All we have to do is to find the largest number λ such $H-\lambda\rho$ is a positive operator. This requires that we solve equation (B) with the modification $\omega^2 \to \omega^2+\lambda$, and check if a solution that satisfies the boundary condition on the left (B2), does not vanish inside the plasma. If it does, we take λ smaller. The first zero of the solution in the plasma, moves monotonically (with λ) toward the boundary r = R, and when it reaches it, λ attains the value of λ_ω. This procedure holds as long as λ_ω is smaller than $\lambda_c = \min_r(\frac{\beta}{\rho} h^2-\sigma^2)$ as described in section V.2. In this case λ_ω is indeed the lowest eigenvalue of H_ω and we know that $\frac{\partial^2}{\partial\omega^2}\lambda_\omega < 0$. This property will be of great help in locating the unique optimal ω.

Appendix B

All Eulerian perturbations can be described as linear combinations of the three perturbed quantities p_*, u, u' defined in section I.2. We write this relation as

$$B_r = \frac{H}{\sigma} u \quad .$$

$$i\rho\sigma^2 (H^2 - \rho\sigma^2) \cdot \underline{\eta} = B\underline{\phi}$$

where

$$\underline{\eta} = (u_\theta, u_z, B_\theta, B_z, \rho)$$

$$\underline{\phi} = (ip_*, u, u')$$

are the perturbed quantities, and B is a 5×3 matrix whose entries depend on the equilibrium profiles.

$$B_{11} = \sigma\left(\frac{m}{r}\rho\sigma^2 - K^2 H B_\theta\right)$$

$$B_{12} = \frac{1}{\sigma}\left(\frac{m}{r}B_\theta H - \rho\sigma^2\right)\left[(H^2 - \rho\sigma^2)v' - (H^2 + \rho\sigma^2)\frac{v}{r}\right] + \frac{1}{\sigma} k B_\theta H (H^2 - \rho\sigma^2) w'$$

$$+ \frac{1}{r} H B_\theta \left(\frac{m}{r}B_\theta H - k B_z H - \rho\sigma^2\right)$$

$$B_{13} = -B_\theta H (H^2 - \rho\sigma^2)$$

$$B_{21} = \sigma(k\rho\sigma^2 - K^2 H B_z)$$

$$B_{22} = \frac{1}{\sigma}\frac{m}{r} HB_z[(H^2-\rho\sigma^2)v'-(H^2+\rho\sigma^2)\frac{v}{r}] + \frac{1}{\sigma}(kB_zH-\rho\sigma^2)(H^2-\rho\sigma^2)w'$$

$$+ \frac{1}{r} HB_z(\frac{m}{r}B_\theta H-kB_zH+\rho\sigma^2)$$

$$B_{23} = -B_zH(H^2-\rho\sigma^2)$$

$$B_{31} = \rho\sigma^2(\frac{m}{r} H-K^2B_\theta)$$

$$B_{32} = \rho B_\theta(H^2-\rho\sigma^2)[\frac{m}{r}(v'-\frac{v}{r})+kw'] + \frac{2}{r} kB_z\rho^2\sigma^2v$$

$$-\rho\sigma(H^2-\rho\sigma^2)B_\theta' - \frac{2}{r}\rho\sigma HkB_\theta B_z$$

$$B_{33} = -\rho\sigma B_\theta(H^2-\rho\sigma^2)$$

$$B_{41} = \rho\sigma^2(kH-K^2B_z)$$

$$B_{42} = \frac{m}{r}B_z\rho[(H^2-\rho\sigma^2)v'-(H^2+\rho\sigma^2)\frac{v}{r}] + kB_z\rho(H^2-\rho\sigma^2)w'$$

$$-\rho\sigma(H^2-\rho\sigma^2)B_z' + \frac{1}{r}\rho\sigma B_z(\frac{m}{r}B_\theta H-kB_zH + \rho\sigma^2)$$

$$B_{43} = -\rho\sigma B_z(H^2-\rho\sigma^2)$$

$$B_{51} = \rho K^2(H^2-\rho\sigma^2)$$

$$B_{52} = \rho(H^2-\rho\sigma^2)[\rho(\frac{m}{r} v' + \frac{mv}{r^2} +kw') - \frac{1}{r\sigma}(\frac{m}{r} B_\theta H-kB_zH + \rho\sigma^2)$$

$$- \frac{\sigma'}{\sigma^2} H^2 - \rho'\sigma)]$$

$$B_{53} = \frac{\rho}{\sigma}(H^2-\rho\sigma^2)^2$$

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